Exploration 2 Euler's Formula

1. $$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \cdots + \frac{(ix)^n}{n!} + \cdots$$
$$= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \cdots + (i)^n \frac{x^n}{n!} + \cdots$$

2. If we isolate the terms in the series that have i as a factor, we get:
$$e^{ix} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \cdots + (i)^n \frac{x^n}{n!} + \cdots$$
$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \left(\frac{x^{2n+1}}{(2n+1)!}\right) + \cdots\right)$$
$$= \cos x + i \sin x.$$
(We are assuming here that we can rearrange the terms of a convergent series without affecting the sum. It happens to be true in this case, but we will see in Section 10.5 that it is not always true.)

3. $e^{i\pi} = \cos \pi + i \sin \pi = -1 + 0 = -1$
Thus, $e^{i\pi} + 1 = 0$

Quick Review 10.3

1. Since $|f(x)| = |2\cos(3x)| \le 2$ on $[-2\pi, 2\pi]$ and $f(0) = 2$, $M = 2$.

2. Since $f(x)$ is increasing and positive on [1, 2], $M = f(2) = 7$.

3. Since $f(x)$ is increasing and positive on [–3, 0], $M = f(0) = 1$.

4. Since the minimum value of $f(x)$ and the maximum value of $f(x)$ is $f(1) = \frac{1}{2}$, $M = \frac{1}{2}$.

5. On [–3, 1], the minimum value of $f(x)$ and the maximum value of $f(x)$ is $f(0) = 2$. On (1, 3], f is increasing and positive, so the maximum value of f is $f(3) = 5$. Thus, $|f(x)| \le 7$ on [–3, 3] and $M = 7$.

6. Yes, since each expression for an nth derivative given by the Quotient Rule will be a rational function whose denominator is a power of $x + 1$.

7. No, since the function $f(x) = |x^2 - 4|$ has a corner at $x = 2$.

8. Yes, since the derivatives of all orders for $\sin x$ and $\cos x$ are defined for all values of x.

9. Yes, since the function $f(x) = e^{-x}$ has derivatives of the form $f^{(n)}(x) = -e^{-x}$ for odd values of n and $f^{(n)}(x) = e^{-x}$ for even values of n, and both of these expressions are defined for all values of x.

10. No, since $f(x) = x^{3/2}$, we have $f'(x) = \frac{3}{2}x^{1/2}$ and $f''(x) = \frac{3}{4}x^{-1/2}$, so $f''(0)$ is undefined.

Section 10.3 Exercises

1. $f(0) = e^{-2x}\Big|_{x=0} = 1$

$f'(0) = -2e^{-2x}\Big|_{x=0} = -2$

$f''(0) = 4e^{-2x}\Big|_{x=0} = 4$, so $\dfrac{f''(0)}{2!} = 2$

$f'''(0) = -8e^{-2x}\Big|_{x=0} = -8$,

so $\dfrac{f'''(0)}{3!} = -\dfrac{4}{3}$

$f^{(4)}(0) = 16e^{-2x}\Big|_{x=0} = 16$,

so $\dfrac{f^{(4)}(0)}{4!} = \dfrac{2}{3}$

$P_4(x) = 1 - 2x + 2x^2 - \dfrac{4}{3}x^3 + \dfrac{2}{3}x^4$

$f(0.2) = P_4(0.2) = 0.6704$

2. $f(0) = \cos\dfrac{\pi x}{2}\Big|_{x=0} = 1$

$f'(0) = -\dfrac{\pi}{2}\sin\dfrac{\pi x}{2}\Big|_{x=0} = 0$

$f''(0) = -\dfrac{\pi^2}{4}\cos\dfrac{\pi x}{2}\Big|_{x=0} = -\dfrac{\pi^2}{4}$,

so $\dfrac{f''(0)}{2!} = -\dfrac{\pi^2}{8}$

$f'''(0) = \dfrac{\pi^3}{8}\sin\dfrac{\pi x}{2}\Big|_{x=0} = 0$, so $\dfrac{f'''(0)}{3!} = 0$

$f^{(4)}(0) = \dfrac{\pi^4}{16}\cos\dfrac{\pi x}{2}\Big|_{x=0} = \dfrac{\pi^4}{16}$,

so $\dfrac{f^{(4)}(0)}{4!} = \dfrac{\pi^4}{384}$

$P_4(x) = 1 - \dfrac{\pi^2}{8}x^2 + \dfrac{\pi^4}{384}x^4$

$f(0.2) = P_4(0.2) \approx 0.9511$

3. $f(0) = 5\sin(-x)\big|_{x=0} = -5\sin x\big|_{x=0} = 0$

$f'(0) = -5\cos x\big|_{x=0} = -5$

$f''(0) = 5\sin x\big|_{x=0} = 0$, so $\dfrac{f''(0)}{2!} = 0$

$f'''(0) = 5\cos x\big|_{x=0} = 5$, so $\dfrac{f'''(0)}{3!} = \dfrac{5}{6}$

$f^{(4)}(0) = -5\sin x\big|_{x=0} = 0$, so $\dfrac{f^{(4)}(0)}{4!} = 0$

$P_4(x) = -5x + \dfrac{5}{6}x^3$

$f(0.2) \approx P_4(0.2) = -\dfrac{149}{150} \approx -0.9933$

4. Substituting x^2 for x in the Maclaurin series given for $\ln(1+x)$ at the end of Section 10.2, we have

$\ln(1+x^2)$

$= x^2 - \dfrac{(x^2)^2}{2} + \dfrac{(x^2)^3}{3} - \cdots + (-1)^{n-1}\dfrac{(x^2)^n}{n} + \cdots$

$= x^2 - \dfrac{x^4}{2} + \dfrac{x^6}{3} - \cdots + (-1)^{n-1}\dfrac{x^{2n}}{n} + \cdots$

Therefore, $P_4(x) = x^2 - \dfrac{x^4}{2}$ and

$f(0.2) \approx P_4(0.2) = 0.0392.$

5. $f(0) = (1-x)^{-2}\Big|_{x=0} = 1$

$f'(0) = 2(1-x)^{-3}\Big|_{x=0} = 2$

$f''(0) = 6(1-x)^{-4}\Big|_{x=0} = 6$, so $\dfrac{f''(0)}{2!} = 3$

$f'''(0) = 24(1-x)^{-5}\Big|_{x=0} = 24$,

so $\dfrac{f'''(0)}{3!} = 4$

$f^{(4)}(0) = 120(1-x)^{-6}\Big|_{x=0} = 120$,

so $\dfrac{f^{(4)}(0)}{4!} = 5$

$P_4(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4$

$f(0.2) \approx P_4(0.2) = 1.56$

6. $\sin x - x + \dfrac{x^3}{3!} = \left(x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots + (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} + \cdots\right) - x + \dfrac{x^3}{3!}$

$$= \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

Note: By replacing n with $n + 2$, the general term can be written as $(-1)^n \dfrac{x^{2n+5}}{(2n+5)!}$

7. $xe^x = x\left(1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^n}{n!} + \cdots\right)$

$$= x + x^2 + \frac{x^3}{2!} + \cdots + \frac{x^{n+1}}{n!} + \cdots$$

8. $\cos^2 x = \dfrac{1}{2} + \dfrac{1}{2}\cos\ (2x)$

$$= \frac{1}{2} + \frac{1}{2}\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \cdots + (-1)^n \frac{(2x)^{2n}}{(2n)!} + \cdots\right)$$

$$= 1 - \frac{4x^2}{2\cdot 2!} + \frac{16x^4}{2\cdot 4!} - \cdots + (-1)^n \frac{2^{2n}x^{2n}}{2\cdot(2n)!} + \cdots$$

$$= 1 - x^2 + \frac{x^4}{3} - \cdots + (-1)^n \frac{2^{2n-1}x^{2n}}{(2n)!} + \cdots$$

9. $\sin^2 x = \dfrac{1}{2} - \dfrac{1}{2}\cos(2x)$

$$= \frac{1}{2} - \frac{1}{2}\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots + (-1)^n \frac{(2x)^{2n}}{(2n)!} + \cdots\right)$$

$$= \frac{4x^2}{2\cdot 2!} - \frac{16x^4}{2\cdot 4!} + \frac{64x^6}{2\cdot 6!} - \cdots + (-1)^{n-1} \frac{2^{2n}x^{2n}}{2\cdot(2n)!} + \cdots$$

$$= x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \cdots + (-1)^{n-1} \frac{2^{2n-1}x^{2n}}{(2n)!} + \cdots$$

Note: By replacing n with n+1, the general term can be written as $(-1)^n \dfrac{2^{2n+1}x^{2n+2}}{(2n+2)!}$.

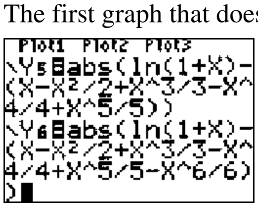
10. $\dfrac{x^2}{1-2x} = x^2\left(\dfrac{1}{1-2x}\right)$

$$= x^2[1 + 2x + (2x)^2 + \cdots + (2x)^n + \cdots]$$

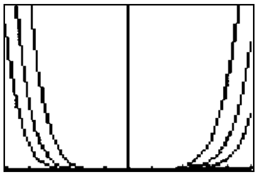
$$= x^2 + 2x^3 + 4x^4 + \cdots + 2^n x^{n+2} + \cdots$$

11. $P_7(x)$. Use the window [–0.5, 0.5] by [0, 0.001] to graph $\mathrm{Y}_n = |\ln(1+x) - P_n(x)|$, n = 1, 2, 3, ...

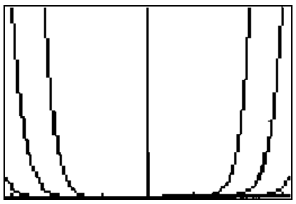
The first graph that does not intersect the top of the window is $\mathrm{Y}_7 = |\ln(1+x) - P_7(x)|$.



Formulas for Y5 and Y6


Graphs for Y5 and Y6 and Y7

12. $P_{12}(x)$. Use the window $[-\pi, \pi]$ by $[0, 0.001]$ to graph $|\cos(x) - P_n(x)|$, $n = 2, 4, 6, \ldots$. The first graph that does not intersect the top of the window is $|\cos(x) - P_{12}(x)|$.


Graphs of $|\cos(x) - P_8(x)|$, $|\cos(x) - P_{10}(x)|$ and $|\cos(x) - P_{12}(x)|$

13. $f(x) = \dfrac{1}{1-2x} = 1 + 2x + (2x)^2 + \cdots$

$|f(x) - P_6(x)| = |(2x)^7 + (2x)^8 + \cdots| = \left|\dfrac{(2x)^7}{1-2x}\right|$

For all $x \in \left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$, $1 - 2x > 0$. Therefore the truncation error is: $\dfrac{(2|x|)^7}{1-2x}$

14. $\left|\dfrac{1}{1-x} - P_9(x)\right| = |x^{10} + x^{11} + \cdots| = \left|\dfrac{x^{10}}{1-x}\right|$

For all $x \in (-1, 1)$, $1 - x > 0$. Therefore the truncation error is: $\dfrac{x^{10}}{1-x}$

15. If $M = 1$ and $r = 1$, then $|\cos^{(n+1)}(t)| \le Mr^{n+1} = 1$ for all t, since all derivatives of the cosine function are sine or cosine functions bounded between -1 and 1.

16. If $M = 1$ and $r = 5$, then $|f^{(n+1)}(t)| \le Mr^{n+1} = 5^{n+1}$ for all t, since the $(n + 1)$st derivative of f is a sine or cosine function (bounded between -1 and 1) times 5^{n+1} (the result of $n+1$ applications of the Chain Rule).

17. If $M = 1$ and $r = 8$ then $|f^{(n+1)}(t)| \le Mr^{n+1} = 8^{n+1}$ for all t, since the $(n + 1)$st derivative of f is a sine or cosine function (bounded between -1 and 1) times 8^{n+1} (the result of $n+1$ applications of the Chain Rule).

18. Note that $|f^{(n+1)}(t)| = |e^{5t} \cdot 5^{n+1}|$, which can be bounded for all t between 0 and x by $|e^{5x} \cdot 5^{n+1}|$ when $x > 0$ and by $|e^0 \cdot 5^{n+1}|$ when $x < 0$. Thus, let M be the larger of the two numbers $\{e^{5x}, 1\}$ and let $r = 5$.

19. Let $f(x) = \sin x$. Then

$P_4(x) = P_3(x) = x - \dfrac{x^3}{6}$, so we use the Remainder Estimation Theorem with $n = 4$. (We use P_4 instead of P_3 because it gives us a better range of x's.) Since $|f^{(5)}(x)| = |\cos x| \le 1$ for all x, we may use $M = r = 1$, giving $|R_4(x)| \le \dfrac{|x|^5}{5!}$, so we may assure that $|R_4(x)| \le 5 \times 10^{-4}$ by requiring $\dfrac{|x|^5}{5!} \le 5 \times 10^{-4}$, or $|x| \le \sqrt[5]{0.06} \approx 0.5697$.
Thus, the absolute error is no greater than 5×10^{-4} when $-0.56 < x < 0.56$ (approximately).
Alternate method: Using graphing techniques, $\left|\sin x - \left(x - \dfrac{x^3}{6}\right) \le 5\times10^{-4}\right|$ when $-0.57 < x < 0.57$.

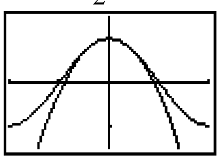
20. Let $f(x) = \cos x$. Then

$P_3(x) = P_2(x) = 1 - \dfrac{x^2}{2}$, so we may use the Remainder Estimation Theorem with $n = 3$. (We use P_3 instead of P_2 because it gives us a better range of x's.) Since $|f^{(4)}(x)| = |\cos x| \le 1$ for all x, we may use $M = r = 1$, giving $|R_3(x)| \le \dfrac{|x|^4}{4!}$. For $|x| < 0.5$, the absolute error

is less than $\frac{(0.5)^4}{4!} \approx 0.0026$ (approximately).
Alternate method: Using graphing techniques, we find that when $|x| < 0.5$,

$$\begin{aligned}|\text{error}| &= \left|\cos x - \left(1 - \frac{x^2}{2}\right)\right| \\ &< \left|\cos 0.5 - \left(1 - \frac{0.5^2}{2}\right)\right| \\ &\approx 0.002583.\end{aligned}$$

The quantity $1 - \frac{x^2}{2}$ tends to be too small, as shown by the graphs of $y = \cos x$ and $y = 1 - \frac{x^2}{2}$.

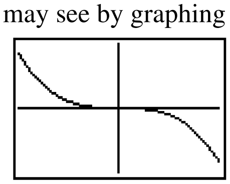


$[-\pi, \pi]$ by $[-1.5, 1.5]$

21. Let $f(x) = \sin x$. Then $P_2(x) = P_1(x) = x$, so we may use the Remainder Estimation Theorem with $n = 2$. Since $|f'''(x)| = |-\cos x| \le 1$ for all x, we may use $M = r = 1$, giving $|R_2(x)| \le \frac{|x|^3}{3!}$. Thus, for $|x| < 10^{-3}$, the maximum possible error is about $\frac{(10^{-3})^3}{3!} \approx 1.67 \times 10^{-10}$.
Alternate method:
Using graphing techniques, we find that when $|x| < 10^{-3}$,

$$\begin{aligned}|\text{error}| &= |\sin x - x| \\ &\le \left|\sin 10^{-3} - 10^{-3}\right| \\ &\approx 1.67 \times 10^{-10}.\end{aligned}$$

The inequality $x < \sin x$ is true for $x < 0$, as we may see by graphing $y = \sin x - x$.



$[-10^{-3}, 10^{-3}]$ by $[-2 \times 10^{-10}, 2 \times 10^{-10}]$

22. Let $f(x) = \sqrt{1+x}$. Then $P_1(x) = 1 + \frac{x}{2}$, so we may use the Remainder Estimation Theorem with $n = 1$. Since $|f''(x)| = \left|-\frac{1}{4}(1+x)^{-3/2}\right|$, which is less than 0.2538 for $|x| < 0.01$, we may use $M = 0.2538$ and $r = 1$, giving $|R_1(x)| \le \frac{0.2538|x|^2}{2!}$. Thus, for $|x| < 0.01$ the maximum possible absolute error is about $\frac{0.2538(0.01)^2}{2!} \approx 1.27 \times 10^{-5}$.
Alternate method:
Using graphing techniques, we find that when $|x| < 0.01$,

$$\begin{aligned}|\text{error}| &= \left|\sqrt{1+x} - \left(1 + \frac{x}{2}\right)\right| \\ &\le \left|\sqrt{1-0.01} - \left(1 - \frac{0.01}{2}\right)\right| \\ &\approx 1.26 \times 10^{-5}.\end{aligned}$$

23. Note that $1 + x + \frac{x^2}{2}$ is the second order Taylor polynomial for $f(x) = e^x$ at $x = 0$, so we may use the Remainder Estimation Theorem with $n = 2$. Since $|f'''(x)| = e^x$, which is less than $e^{0.1}$ when $|x| < 0.1$, we may use $M = e^{0.1}$ and $r = 1$, giving $|R_2(x)| \le \frac{e^{0.1}|x|^3}{3!}$. Thus, for $|x| < 0.1$, the maximum possible error is about $\frac{e^{0.1}(0.1)^3}{3!} \approx 1.842 \times 10^{-4}$.

24. Note that $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$ and $e^{-x} = 1 - x + \frac{x^2}{2!} - \cdots + (-1)^n \frac{x^n}{n!} + \cdots$ Thus the terms with n even will cancel for $\sinh x = \frac{1}{2}(e^x - e^{-x})$, and the terms with n

odd will cancel for $\cosh x = \frac{1}{2}(e^x - e^{-x})$.

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2n+1}}{(2n+1)!} + \ldots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2n}}{(2n)!} + \ldots$$

25. All of the derivatives of cosh x are either cosh x or sinh x. For any real x, cosh x and sinh x are both bounded by $e^{|x|}$. So for any real x, let $M = e^{|x|}$ and $r = 1$ in the Remainder Estimation Theorem. This gives $|R_n(x)| \le \frac{e^{|x|}x^{n+1}}{(n+1)!}$, but for any fixed value of x, $\lim\limits_{n\to\infty} \frac{e^{|x|}|x|^{n+1}}{(n+1)!} = 0$. It follows that the series converges to cosh x for all real values of x.

26. For $n = 0$, Taylor's Theorem with Remainder says that if f has derivatives of all orders in an open interval I containing a, then for each x in I, $f(x) = f(a) + R(x)$, where $R(x) = f'(c)(x-a)$, so $f(x) = f(a) + f'(c)(x-a)$ for some c between a and x. Letting $b = x$, this equation is $f(b) = f(a) + f'(c)(b-a)$, which is equivalent to $f'(c) = \frac{f(b)-f(a)}{b-a}$ for some c between a and b. Thus, for the class of functions that have derivatives of all orders in an open interval containing a and b, the Mean Value Theorem can be considered a special case of Taylor's Theorem.

27. $f(0) = \ln(\cos x)\big|_{x=0} = \ln\ 1 = 0$

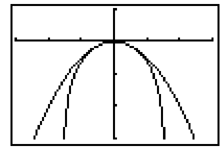
$f'(0) = \frac{1}{\cos x}(-\sin x)\big|_{x=0} = -\tan x\big|_{x=0} = 0$

$f''(0) = -\sec^2 x\Big|_{x=0} = -1$ so $\frac{f''(0)}{2!} = -\frac{1}{2}$

(a) $L(x) = 0$

(b) $P_2(x) = -\frac{1}{2}x^2$

(c) The graphs of the linear and quadratic approximations fit the graph of the function near $x = 0$.



[–3, 3] by [–3, 1]

28. $f(0) = e^{\sin x}\Big|_{x=0} = e^0 = 1$

$f'(0) = e^{\sin x}\cos x\big|_{x=0} = 1$

$f''(0)$

$= \left[(e^{\sin x})(-\sin x) + (\cos x)(e^{\sin x}\cos x)\right]\Big|_{x=0}$

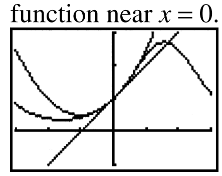
$= 1,$

so $\frac{f''(0)}{2!} = \frac{1}{2}$

(a) $L(x) = 1 + x$

(b) $P_2(x) = 1 + x + \frac{x^2}{2}$

(c) The graphs of the linear and quadratic approximations fit the graph of the function near $x = 0$.



[–3, 3] by [–1,3]

29. $f(0) = (1-x^2)^{-1/2}\big|_{x=0} = 1$

$f'(0) = -\frac{1}{2}(1-x^2)^{-3/2}(-2x)\big|_{x=0}$

$= x(1-x^2)^{-3/2}\big|_{x=0}$

$= 0$

$f''(0)$

$= (x)\left[-\frac{3}{2}(1-x^2)^{-5/2}(-2x)\right] + (1-x^2)^{-3/2}\big|_{x=0}$

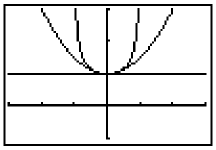
$= 1$

so $\frac{f''(0)}{2!} = \frac{1}{2}$

(a) $L(x) = 1$

(b) $P_2(x) = 1 + \frac{x^2}{2}$

(c) The graphs of the linear and quadratic approximations fit the graph of the function near $x = 0$.



[–3, 3] by [–1, 3]

30. $f(0) = \sec x\big|_{x=0} = 1$

$f'(0) = \sec x \tan x\big|_{x=0} = 0$

$f''(0)$

$= (\sec x)(\sec^2 x) + (\tan x)(\sec x \tan x)\big|_{x=0}$

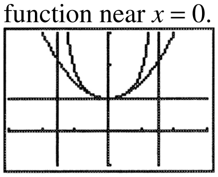
$= 1$

so $\dfrac{f''(0)}{2!} = \dfrac{1}{2}$

(a) $L(x) = 1$

(b) $P_2(x) = 1 + \dfrac{x^2}{2}$

(c) The graphs of the linear and quadratic approximations fit the graph of the function near $x = 0$.



[−3, 3] by [−1, 3]

31. $f(0) = \tan x\big|_{x=0} = 0$

$f'(0) = \sec^2 x\big|_{x=0} = 1$

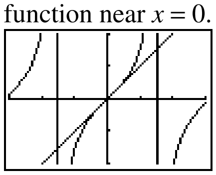
$f''(0) = (2 \sec x)(\sec x \tan x)\big|_{x=0} = 0,$

so $\dfrac{f''(0)}{2!} = 0$

(a) $L(x) = x$

(b) $P_2(x) = x$

(c) The graphs of the linear and quadratic approximations fit the graph of the function near $x = 0$.



[–3, 3] by [–2, 2]

32. $f(0) = (1+x)^k\big|_{x=0} = 1$

$f'(0) = k(1+x)^{k-1}\big|_{x=0} = k$

$f''(0) = k(k-1)(1+x)^{k-2}\big|_{x=0} = k(k-1),$

so $\dfrac{f''(0)}{2!} = \dfrac{k(k-1)}{2}$

$P_2(x) = 1 + kx + \dfrac{k(k-1)}{2}x^2$

For $k = 3$, we have $f(x) = (1+x)^3$ and $f'''(x) = 6$. We may use the Remainder Estimation Theorem with $n = 2$, $M = 6$, and $r = 1$, giving $|R_2(x)| \le \dfrac{6|x|^3}{3!} = |x|^3$. (In this particular case it is actually true that $R_2(x) = x^3$, since $f(x)$ is a cubic polynomial.) Thus the absolute error is less than $\dfrac{1}{100}$ whenever $|x|^3 < 0.01$. In the interval [0, 1], this occurs when $0 \le x < \sqrt[3]{0.01} \approx 0.215$.

Alternate method:

Note that $P_2(x) = 1 + 3x + 3x^2$. Using graphing techniques, $\left|(1+x)^3 - (1+3x+3x^2)\right| < \dfrac{1}{100}$ when $|x| < 0.215$.

33. Let $f(x) = e^x$. Then $P_3(x) = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6}$, so we may use the Remainder Estimation Theorem with $n = 3$. Since $\left|f^{(4)}(x)\right| = e^x$, which is no more than $e^{0.1}$ when $|x| \le 0.1$, we may use $M = e^{0.1}$ and $r = 1$, giving $|R_3(x)| \le \dfrac{e^{0.1}|x|^4}{4!}$. Thus, for $|x| \le 0.1$, the maximum possible absolute error is about $\dfrac{e^{0.1}(0.1)^4}{24} \approx 4.605\times10^{-6}$.

Alternate method:

Using graphing techniques, when $|x| \le 0.1$,

$$\begin{aligned}|\text{error}| &= \left|e^x - \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right)\right| \\ &\le \left|e^{0.1} - \left(1 + 0.1 + \frac{0.01}{2} + \frac{0.001}{6}\right)\right| \\ &\approx 4.251\times10^{-6}.\end{aligned}$$

34. Since the Maclaurin series is $\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots$, $P_3(x) = 1 + x + x^2 + x^3$.

Since $\left|f^{(4)}(x)\right| = 24(1-x)^{-5}$, which is no more than $24(0.9)^{-5}$ when $|x| \le 0.1$, we may use

$M = 24(0.9)^{-5}$ and $r = 1$, giving $|R_3(x)| \le \frac{24(0.9)^{-5}|x|^4}{4!} = \frac{|x|^4}{0.9^5}$. Thus, for $|x| \le 0.1$, an upper bound for the magnitude of the approximation error is $\frac{0.1^4}{0.9^5} \approx 1.694 \times 10^{-4}$. Rounding up to be safe, an upper bound is 1.70×10^{-4}.

Alternate method:

Using graphing techniques, when $|x| \le 0.1$,

$$\begin{aligned}|\text{error}| &= \left|\frac{1}{1-x} - (1 + x + x^2 + x^3)\right| \\ &\le \left|\frac{1}{1-0.1} - 1.111\right| \approx 1.11 \times 10^{-4}.\end{aligned}$$

35. **(a)** No

(b) Yes, since

$$\begin{aligned}\frac{dy}{dx} &= e^{-x^2} \\ &= 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \cdots + \frac{(-x^2)^n}{n!} + \cdots \\ &= 1 - x^2 + \frac{x^4}{2!} - \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots.\end{aligned}$$

The constant term of y is $y(0) = 2$, and we may obtain the remaining terms of y by integrating the above series.

$$y = 2 + x - \frac{x^3}{3} + \frac{x^5}{10} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)n!} + \cdots$$

By substituting $n - 1$ for n, the general term may also be written as $(-1)^{n-1} \frac{x^{2n-1}}{(2n-1)(n-1)!}$.

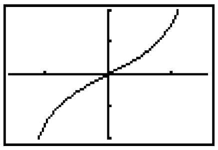
(c) The power series equals the function y for all real values of x. This is because the series for e^{-x^2} converges for all real values of x, so Theorem 2 of Section 10.1 implies that the new series also converges for all x.

36. **(a)** Substitute $-x$ for x in the Maclaurin series for $\ln(1 + x)$ given at the end of Section 10.2.

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^n}{n} - \cdots$$

(b)
$$\begin{aligned}\ln \frac{1+x}{1-x} &= \ln(1+x) - \ln(1-x) \\ &= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots\right) + \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n}\right) \\ &= 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \cdots + \frac{2x^{2n+1}}{2n+1} + \cdots\end{aligned}$$

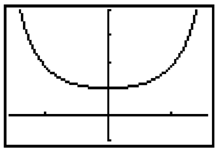
37. **(a)**



$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ by $[-2, 2]$

The series approximates tan x.

(b)



$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ by $[-1, 4]$

The series approximates sec x.

38. False; if $f'(a)$ happens to be 0, the linearization is a constant function.

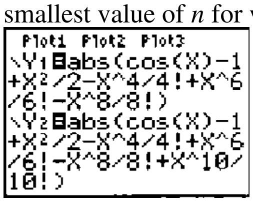
39. True; the coefficient of x is $f'(0)$.

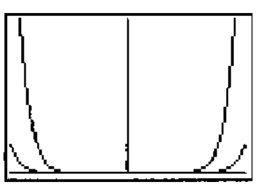
40. D; $1.5 - \dfrac{1.5^3}{3!} + \dfrac{1.5^5}{5!} \approx 1.001$

41. E; coefficient of $x^{12} = \dfrac{f^{(12)}(0)}{12!} = \dfrac{1}{11!}$,

$f^{(12)}(0) = \dfrac{12!}{11!} = 12$

42. B; consider the graphs of $Y_1 = |P_8(x) - \cos(x)|$ and $Y_2 = |P_{10}(x) - \cos(x)|$ in the window $[-\pi, \pi]$ by $[0, 0.01]$. The graph of y_1 intersects the top of the window, but the graph of y_{10} does not, so 10 is the smallest value of n for which the maximum possible error on the interval $[-\pi, \pi]$ is less than 0.01.





43. A

44. **(a)** $\sin^2(x) = \frac{1}{2}(1-\cos 2x)$

$$= \frac{1}{2} - \frac{1}{2}\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots + (-1)^n \frac{(2x)^{2n}}{(2n)!} + \cdots\right)$$
$$= \frac{4x^2}{2\cdot 2!} - \frac{16x^4}{2\cdot 4!} + \frac{64x^6}{2\cdot 6!} - \frac{256x^8}{2\cdot 8!} + \cdots - (-1)^n \frac{2^{2n}x^{2n}}{2\cdot(2n)!} + \cdots$$
$$= x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \frac{x^8}{315} + \cdots + (-1)^{n+1}\frac{2^{2n-1}x^{2n}}{(2n)!} + \cdots$$

(b) $$2\sin x\cos x = \frac{d}{dx}\left(x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \frac{x^8}{315} + \cdots + (-1)^{n+1}\frac{2^{2n-1}x^{2n}}{(2n)!} + \cdots\right)$$
$$= 2x - \frac{4x^3}{3} + \frac{12x^5}{45} - \frac{8x^7}{315} + \cdots + (-1)^{n+1}\frac{2^{2n-1}\cdot 2n\cdot x^{2n-1}}{(2n)!} + \cdots$$
$$= 2x - \frac{4x^3}{3} + \frac{4x^5}{15} - \frac{8x^7}{315} + \cdots + (-1)^{n+1}\frac{2^{2n-1}\cdot x^{2n-1}}{(2n-1)!} + \cdots$$

(c) Writing out the first few terms of the series for sin $(2x)$ and simplifying will show that the beginning terms match. The tricky part is matching up the general terms. It is helpful to take the series we got in part (b) and carry it out to the $(n+1)^{\text{st}}$ term:

$$2\sin x\cos x = 2x - \frac{4x^3}{3} + \cdots + (-1)^{n+1}\frac{2^{2n-1}\cdot x^{2n-1}}{(2n-1)!} + (-1)^{(n+1)+1}\frac{2^{2(n+1)-1}\cdot x^{2(n+1)-1}}{(2(n+1)-1)!} + \cdots$$
$$= 2x - \frac{4x^3}{3} + \cdots + (-1)^{n+1}\frac{2^{2n-1}\cdot x^{2n-1}}{(2n-1)!} + (-1)^{n+2}\frac{2^{2n+1}\cdot x^{2n-1}}{(2n+1)!} + \cdots$$
$$= 2x - \frac{4x^3}{3} + \cdots + (-1)^{n-1}\frac{(2x)^{2n-1}}{(2n-1)!} + (-1)^n\frac{(2x)^{2n+1}}{(2n+1)!} + \cdots$$
$$= \sin 2x$$

45. **(a)** It works. For example, let $n = 2$. Then $P = 3.14$ and $P + \sin P \approx 3.141592653$, which is accurate to more than 6 decimal places.

(b) Let $P = \pi + x$ where x is the error in the original estimate. Then

$$P + \sin P = (\pi + x) + \sin(\pi + x)$$
$$= \pi + x - \sin x$$

But by the Remainder Theorem, $|x - \sin x| < \frac{|x|^3}{6}$. Therefore, the difference between the new estimate $P + \sin P$ and π is less than $\frac{|x|^3}{6}$.

46. **(a)** $$\frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{(\cos\theta + i\sin\theta) + (\cos(-\theta) + i\sin(-\theta))}{2}$$
$$= \frac{\cos\theta + i\sin\theta + \cos\theta - i\sin\theta}{2}$$
$$= \frac{2\cos\theta}{2}$$
$$= \cos\theta$$

(b) $$\frac{e^{i\theta}-e^{-i\theta}}{2i}=\frac{(\cos\theta+i\sin\theta)-(\cos(-\theta)+i\sin(-\theta))}{2i}$$
$$=\frac{(\cos\theta+i\sin\theta)-(\cos\theta-i\sin\theta)}{2i}$$
$$=\frac{2i\sin\theta}{2i}$$
$$=\sin\theta$$

47. Note that $f(-1)=\frac{1}{2}$. The partial sums of the Maclaurin series $\sum_{n=0}^{\infty}(-1)^n$ are 1, 0, 1, 0, 1, and so on, so the remainders are $-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}$, and so on. Thus $R_n(-1)=\frac{(-1)^{n+1}}{2}$, and $\lim_{n\to\infty}\left(\frac{(-1)^{n+1}}{2}\right)\neq 0.$

48. First, note that $f(x)=(1-x)^{-1}$, so repeated differentiation gives $f^{(n+1)}(x)=(n+1)!(1-x)^{-(n+2)}$. Therefore

$$R_n(-1)=f^{(n+1)}(c)\frac{(-1-0)^{n+1}}{(n+1)!}$$
$$=(n+1)!(1-c)^{-(n+2)}\frac{(-1)^{n+1}}{(n+1)!}$$
$$=\frac{(-1)^{n+1}}{(1-c)^{n+2}},\text{ where } -1<c<0.$$

From Exercise 47, we have $R_n(-1)=\frac{(-1)^{n+1}}{2}$, so $\frac{(-1)^{n+1}}{2}=\frac{(-1)^{n+1}}{(1-c)^{n+2}}$. Solving, we find that $(1-c)^{n+2}=2$, from which we get $c=1-2^{\left(\frac{1}{n+2}\right)}$. (Note that, indeed, $-1<c<0$.)

49. The derivative of the right-hand side is

$$\frac{a-ib}{a^2+b^2}(a+ib)e^{(a+ib)x}=\frac{a^2-(ib)^2}{a^2+b^2}e^{(a+ib)x}$$
$$=\frac{a^2+b^2}{a^2+b^2}e^{(a+ib)x}$$
$$=e^{(a+ib)x},$$

which confirms the antiderivative formula.

50. $$\int e^{ax}\cos bx\,dx+i\int e^{ax}\sin bx\,dx=\int e^{(a+ib)x}\,dx$$
$$=\frac{a-ib}{a^2+b^2}e^{(a+ib)x}$$
$$=\frac{a-ib}{a^2+b^2}e^{ax}(\cos bx+i\sin bx)$$
$$=\left(\frac{e^{ax}}{a^2+b^2}\right)(a\cos bx+b\sin bx-ib\cos bx+ia\sin bx)$$
$$=\left(\frac{e^{ax}}{a^2+b^2}\right)\big[(a\cos bx+b\sin bx)+i(a\sin bx-b\cos bx)\big]$$

Separating the real and imaginary parts gives $\int e^{ax}\cos bx\,dx=\frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)$ and

$\int e^{ax}\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx).$

Quick Quiz Sections 10.1–10.3

1. D; $\displaystyle\sum_{n=0}^{\infty}\frac{\pi^n}{e^{2n}}=\sum_{n=0}^{\infty}\left(\frac{\pi}{e^2}\right)^n=\frac{1}{1-\frac{\pi}{e^2}}=\frac{e^2}{e^2-\pi}$

2. A; $2-1x+\dfrac{6x^2}{2!}+\dfrac{12x^3}{3!}=2-x+3x^2+2x^3$

3. E; $\dfrac{1}{x}=\dfrac{1}{1+(x-1)}$
Substitute $(x-1)$ into the Maclaurin series for $\dfrac{1}{1+x}$ to obtain the series: $\displaystyle\sum_{n=0}^{\infty}(-1)^n(x-1)^n$

4. **(a)** Since the series is geometric, it converges if and only if $|r|<1$, where $r=\dfrac{x+2}{3}$.
$\left|\dfrac{x+2}{3}\right|<1\Rightarrow|x+2|<3\Rightarrow-5<x<1.$ The interval of convergence is (–5, 1).

(b) The series is geometric with first term 2 and common ratio $r=\dfrac{x+2}{3}$. It therefore converges to $\dfrac{2}{1-\frac{x+2}{3}}=\dfrac{6}{1-x}$.

Section 10.4 Radius of Convergence (pp. 507–516)

Exploration 1 Finishing the Proof of the Ratio Test

1. For $\displaystyle\sum\frac{1}{n}$: $L=\displaystyle\lim_{n\to\infty}\frac{\frac{1}{n+1}}{\frac{1}{n}}=\lim_{n\to\infty}\frac{n}{n+1}=1.$

For $\displaystyle\sum\frac{1}{n^2}$: $L=\displaystyle\lim_{n\to\infty}\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}}=\lim_{n\to\infty}\frac{n^2}{(n+1)^2}=1.$

2. **(a)** $\displaystyle\int_1^{\infty}\frac{1}{x}\,dx=\lim_{k\to\infty}\left(\ln x\Big|_1^k\right)=\lim_{k\to\infty}\ln k=\infty.$

(b) $\displaystyle\int_1^{\infty}\frac{1}{x^2}\,dx=\lim_{k\to\infty}\left(-x^{-1}\Big|_1^k\right)$
$\displaystyle=\lim_{k\to\infty}\left(-\frac{1}{k}+1\right)$
$=1.$

3. Figure 10.14a shows that $\sum\frac{1}{n}$ is greater than $\displaystyle\int_1^{\infty}\frac{1}{x}\,dx.$ Since the integral diverges, so must the series.

Figure 10.14b shows that $\sum\frac{1}{n^2}$ is less than $\displaystyle 1+\int_1^{\infty}\frac{1}{x^2}\,dx.$
Since the integral converges, so must the series.

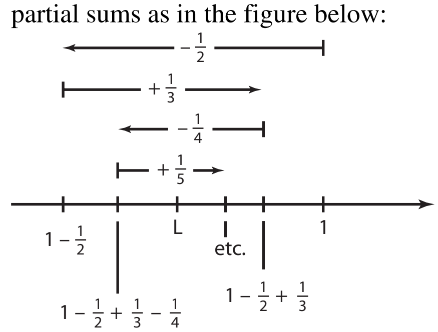
4. These two examples prove that $L=1$ can be true for either a divergent series or a convergent series. The Ratio Test itself is therefore inconclusive when $L=1$.

Exploration 2 Revisiting a Maclaurin Series

1. $\displaystyle L=\lim_{n\to\infty}\frac{|x|^{n+1}}{n+1}\cdot\frac{n}{|x|^n}$
$\displaystyle=\lim_{n\to\infty}\frac{n}{n+1}|x|$
$=|x|.$
The series converges absolutely when $|x|<1$, so the radius of convergence is 1.

2. When $x=-1$, the series becomes
$-1-\dfrac{1}{2}-\dfrac{1}{3}-\cdots-\dfrac{1}{n}-\cdots.$
Each term in this series is the negative of the corresponding term in the divergent series of Figure 10.14a. Just as $\sum\frac{1}{n}$ diverges to $+\infty$, this series diverges to $-\infty$.

3. Geometrically, we chart the progress of the partial sums as in the figure below:



4. The series converges at the right-hand endpoint. As shown in the picture above, the partial sums are closing in on some limit L as they oscillate left and right by constantly decreasing amounts.

5. We know that the series does not converge absolutely at the right-hand endpoint, because $\sum \frac{1}{n}$ diverges (Exploration 1 of this section).

Quick Review 10.4

1. $\lim_{n\to\infty} \frac{n|x|}{n+1} = |x| \lim_{n\to\infty} \frac{n}{n+1} = |x|$

2. $$\lim_{n\to\infty} \frac{n^2|x-3|}{n(n-1)} = |x-3| \cdot \lim_{n\to\infty} \frac{n^2}{n^2-n}$$
$$= |x-3| \cdot \lim_{n\to\infty} \frac{1}{1-\frac{1}{n}}$$
$$= |x-3|$$

3. $\lim_{n\to\infty} \frac{|x|^n}{n!} = 0$
(Note: This limit is similar to the limit which is discussed at the end of Example 3 in Section 10.3.)

4. $$\lim_{n\to\infty} \frac{(n+1)^4 x^2}{(2n)^4}$$
$$= x^2 \lim_{n\to\infty} \frac{n^4+4n^3+6n^2+4n+1}{16n^4}$$
$$= x^2\left(\frac{1}{16}\right)$$
$$= \frac{x^2}{16}$$

5. $\lim_{n\to\infty} \frac{|2x+1|^{n+1} 2^n}{2^{n+1}|2x+1|^n} = \lim_{n\to\infty} \frac{|2x+1|}{2} = \frac{|2x+1|}{2}$

6. Since $n^2 > 5n$ implies $n > 5$, take
$a_n = n^2, b_n = 5n,$ and $N = 6.$
Then $a_n > b_n$ for all $n \ge N.$

7. Clearly, $a_5 = b_5$. (They both equal 5^5.)
Using a calculator to graph the functions x^5 and 5^x, we see that $5^x > x^5$ for $x > 5$.
Take $a_n = 5^n, b_n = n^5$ and $N = 6$.
Then $a_n > b_n$ for all $n \ge N$.

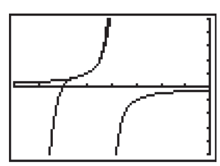
8. Since $0 < 1$, $\ln n < \sqrt{n}$ for $n = 1$. Using a calculator to graph the functions $\ln x$ and $\sqrt{x}$, we see that $\sqrt{x} > \ln x$ for $x > 1$.
Take $a_n = \sqrt{n}, b_n = \ln n,$ and $N = 1$. Then $a_n > b_n$ for all $n \ge N$.

9. Using a calculator, we find that $10^{24} > 24!$, but $10^{25} < 25!$, so $\frac{1}{10^{25}} < \frac{1}{25!}$. Graphing the functions $\frac{1}{10^x}$ and $\frac{1}{x!}$, we see that $\frac{1}{10^x} > \frac{1}{x!}$ for $x > 25$. Take $a_n = \frac{1}{10^n}$, $b_n = \frac{1}{n!}$, and $N = 25$. Then $a_n > b_n$ for all $n \ge N$.

10. Since $n^2 < n^3$ for all $n > 1$, it follows that $\frac{1}{n^2} > \frac{1}{n^3} = n^{-3}$ for $n \ge 2$. Take
$a_n = \frac{1}{n^2}, b_n = n^{-3},$ and $N = 2$. Then $a_n > b_n$ for all $n \ge N$.

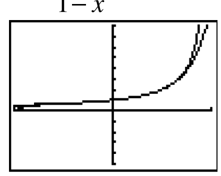
Section 10.4 Exercises

1. $-6 < x < -4$; the graph of the function $y = \frac{-1}{x+4}$ and $P_5(x)$ illustrates the support.



[–8, 0] by [–5, 5]

2. $-1 < x < 1$; The graph of the function $y = \frac{1}{1-x}$ and $P_9(x)$ illustrates the support.



[–1, 1] by [–5, 8]

3. $\frac{x^{3n}}{2n!+1} \le \frac{x^{3n}}{n!} \le \frac{(x^3)^n}{n!}$ and $\sum_{n=0}^{\infty} \frac{(x^3)^n}{n!}$ is the Taylor series for e^{x^3} which converges for all x.

4. $\frac{x^{2n}}{n!+2} \le \frac{x^{2n}}{n!} \le \frac{(x^2)^n}{n!}$ and $\sum_{n=0}^{\infty} \frac{(x^2)^n}{n!}$ is the Taylor series for e^{x^2} which converges for all x.

5. $\left|\frac{(\cos x)^n}{n!+1}\right| \le \frac{|\cos x|^n}{n!} \le \frac{1}{n!}$ and $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges to e.

6. $\left|\frac{2(\sin x)^n}{n!+3}\right| \le \frac{2|\sin x|^n}{n!} \le \frac{2}{n!}$ and $\sum_{n=0}^{\infty} \frac{2}{n!}$ converges to $2e$.

7. This is a geometric series which converges only for $|x| < 1$, so the radius of convergence is 1.

8. This is a geometric series which converges only for $|x+5| < 1$, so the radius of convergence is 1.

9. This is a geometric series which converges only for $|-(4x+1)| < 1$, or $\left|x+\frac{1}{4}\right| < \frac{1}{4}$, so the radius of convergence is $\frac{1}{4}$.

10. $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|3x-2|^{n+1}}{n+1}\cdot\frac{n}{|3x-2|^n} = |3x-2|$$

The series converges for $|3x-2| < 1$, or $\left|x-\frac{2}{3}\right| < \frac{1}{3}$, and diverges for $\left|x-\frac{2}{3}\right| > \frac{1}{3}$, so the radius of convergence is $\frac{1}{3}$.

11. This is a geometric series which converges only for $\left|\frac{x-2}{10}\right| < 1$, or $|x-2| < 10$, so the radius of convergence is 10.

12. $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)|x|^{n+1}}{n+3}\cdot\frac{n+2}{n|x|^n} = |x|\cdot\lim_{n\to\infty}\frac{n^2+3n+2}{n^2+3n} = |x|$$

The series converges for $|x| < 1$ and diverges for $|x| > 1$, so the radius of convergence is 1.

13. $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x|^{n+1}}{(n+1)\sqrt{n+1}\;3^{n+1}}\cdot\frac{n\sqrt{n}3^n}{|x|^n} = \frac{|x|}{3}\cdot\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{3/2} = \frac{|x|}{3}$$

The series converges for $|x| < 3$ and diverges for $|x| > 3$, so the radius of convergence is 3.

14. $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x|^{2n+3}}{(n+1)!}\cdot\frac{n!}{|x|^{2n+1}} = \lim_{n\to\infty}\frac{x^2}{n+1} = 0$$

The series converges for all values of x, so the radius of convergence is ∞.

15. $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)|x+3|^{n+1}}{5^{n+1}}\cdot\frac{5^n}{n|x+3|^n} = \frac{|x+3|}{5}\cdot\lim_{n\to\infty}\frac{n+1}{n} = \frac{|x+3|}{5}$$

The series converges for $|x+3| < 5$ and diverges for $|x+3| > 5$, so the radius of convergence is 5.

16. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$
$$= \lim_{n\to\infty}\frac{(n+1)|x|^{n+1}}{4^{n+1}[(n+1)^2+1]}\cdot\frac{4^n(n^2+1)}{n|x|^n}$$
$$= \frac{|x|}{4}\cdot\lim_{n\to\infty}\frac{n^3+n^2+n+1}{n^3+2n^2+2n}$$
$$= \frac{|x|}{4}$$
The series converges for $|x| < 4$ and diverges for $|x| > 4$, so the radius of convergence is 4.

17. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)!|x-4|^{n+1}}{n!|x-4|^n}$
$$= \lim_{n\to\infty}(n+1)|x-4|$$
$$= \infty \ (x \neq 4)$$
The series converges only for $x = 4$, so the radius of convergence is 0.

18. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\sqrt{n+1}|x|^{n+1}}{3^{n+1}}\cdot\frac{3^n}{\sqrt{n}|x|^n}$
$$= \frac{|x|}{3}\cdot\lim_{n\to\infty}\sqrt{\frac{n+1}{n}}$$
$$= \frac{|x|}{3}$$
The series converges for $|x| < 3$ and diverges for $|x| > 3$, so the radius of convergence is 3.

19. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\left|(-2)^{n+1}\right|(n+2)|x-1|^{n+1}}{\left|(-2)^n\right|(n+1)|x-1|^n}$
$$= 2|x-1|\cdot\lim_{n\to\infty}\frac{n+2}{n+1}$$
$$= 2|x-1|$$
The series converges for $|x-1| < \frac{1}{2}$ and diverges for $|x-1| > \frac{1}{2}$, so the radius of convergence is $\frac{1}{2}$.

20. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|4x-5|^{2n+3}}{(n+1)^{3/2}}\cdot\frac{n^{3/2}}{|4x-5|^{2n+1}}$
$$= (4x-5)^2\cdot\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{3/2}$$
$$= (4x-5)^2$$
The series converges for $(4x-5)^2 < 1$, which is equivalent to $|4x-5| < 1$, or $\left|x-\frac{5}{4}\right| < \frac{1}{4}$ and diverges for $\left|x-\frac{5}{4}\right| > \frac{1}{4}$. The radius of convergence is $\frac{1}{4}$.

21. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x+\pi|^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{|x+\pi|^n}$
$$= |x+\pi|\cdot\lim_{n\to\infty}\sqrt{\frac{n+1}{n}}$$
$$= |x+\pi|$$
The series converges for $|x+\pi| < 1$ and diverges for $|x+\pi| > 1$, so the radius of convergence is 1.

22. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$
$$= \lim_{n\to\infty}\left|\frac{\left(x-\sqrt{2}\right)^{2n+3}}{2^{n+1}}\cdot\frac{2^n}{\left(x-\sqrt{2}\right)^{2n+1}}\right|$$
$$= \frac{1}{2}\left(x-\sqrt{2}\right)^2$$
The series converges for $\frac{1}{2}(x-\sqrt{2})^2 < 1$, which is equivalent to $\left|x-\sqrt{2}\right| < \sqrt{2}$, and diverges for $\left|x-\sqrt{2}\right| > \sqrt{2}$. The radius of convergence is $\sqrt{2}$.

23. This is a geometric series with first term $a = 1$ and common ratio $r = \frac{(x-1)^2}{4}$. It converges only when $\left|\frac{(x-1)^2}{4}\right| < 1$, so the interval of convergence is $-1 < x < 3$.

$$\text{Sum} = \frac{a}{1-r} = \frac{1}{1-\frac{(x-1)^2}{4}} = \frac{4}{4-(x-1)^2} = \frac{4}{-x^2+2x+3} = -\frac{4}{x^2-2x-3}$$

24. This is a geometric series with first term $a = 1$ and common ratio $r = \frac{(x+1)^2}{9}$. It converges only when $\left|\frac{(x+1)^2}{9}\right| < 1$, so the interval of convergence is $-4 < x < 2$.

$$\text{Sum} = \frac{a}{1-r} = \frac{1}{1-\frac{(x+1)^2}{9}} = \frac{9}{9-(x+1)^2} = \frac{9}{-x^2-2x+8} = -\frac{9}{x^2+2x-8}$$

25. This is a geometric series with first term $a = 1$ and common ratio $r = \frac{\sqrt{x}}{2} - 1$. It converges only when $\left|\frac{\sqrt{x}}{2} - 1\right| < 1$, so the interval of convergence is $0 < x < 16$.

$$\text{Sum} = \frac{a}{1-r} = \frac{1}{1-\left(\frac{\sqrt{x}}{2}-1\right)} = \frac{2}{4-\sqrt{x}}$$

26. This is a geometric series with first term $a = 1$ and common ratio $r = \ln x$. It converges only when $|\ln x| < 1$, so the interval of convergence is $\frac{1}{e} < x < e$.

$$\text{Sum} = \frac{a}{1-r} = \frac{1}{1-\ln x}$$

27. This is a geometric series with first term $a = 1$ and common ratio $\frac{x^2-1}{3}$. It converges only when $\left|\frac{x^2-1}{3}\right| < 1$, which is equivalent to $-2 < x^2 < 4$. It is always true that $x^2 > -2$, and $x^2 < 4$ implies that $|x| < 2$, so the interval of convergence is $-2 < x < 2$.

$$\text{Sum} = \frac{a}{1-r} = \frac{1}{1-\frac{x^2-1}{3}} = \frac{3}{3-(x^2-1)} = \frac{3}{4-x^2}$$

28. This is a geometric series with first term $a = 1$ and common ratio $\frac{\sin x}{2}$. Since $\left|\frac{\sin x}{2}\right| < 1$ for all x, the interval of convergence is $-\infty < x < \infty$.

$$\text{Sum} = \frac{a}{1-r} = \frac{1}{1-\frac{\sin x}{2}} = \frac{2}{2-\sin x}$$

29. Diverges by the nth-Term Test, since $\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$.

30. Diverges by the nth-Term Test, since $\lim_{n\to\infty} \frac{2^n}{n+1} = \infty$. (The Ratio Test can also be used.)

31. Converges by the Ratio Test, since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2-1}{2^{n+1}} \cdot \frac{2^n}{n^2-1} = \frac{1}{2} < 1$.

32. Converges, because it is a geometric series with $r = \frac{1}{8}$, so $|r| < 1$.

33. Converges by the Ratio Test, since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}}{(3^{n+1}+1)} \cdot \frac{3^n+1}{2^n} = \frac{2}{3} < 1$.

Alternate method: Note that $\frac{2^n}{3^n+1} < \left(\frac{2}{3}\right)^n$ for all n. Since $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$ converges, $\sum_{n=1}^{\infty}\frac{2^n}{3^n+1}$ converges by the Direct Comparison Test.

34. Diverges by the nth-Term Test, since

$$\begin{aligned}\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) &= \lim_{n\to\infty}\frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)}\\ &= \lim_{x\to 0}\frac{\sin x}{x} \qquad \left(\text{where } x = \frac{1}{n}\right)\\ &= 1\\ &\neq 0\end{aligned}$$

35. Converges by the Ratio Test, since

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(n+1)^2 e^{-n-1}}{n^2 e^{-n}} = e^{-1} < 1.$$

36. Converges by the Ratio Test, since

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(n+1)^{10}}{10^{n+1}}\cdot\frac{10^n}{n^{10}} = \frac{1}{10} < 1.$$

37. Converges by the Ratio Test, since

$$\begin{aligned}\lim_{n\to\infty}\frac{a_{n+1}}{a_n} &= \lim_{n\to\infty}\frac{(n+4)!}{3!(n+1)!3^{n+1}}\cdot\frac{3!n!3^n}{(n+3)!}\\ &= \lim_{n\to\infty}\frac{n+4}{3(n+1)}\\ &= \frac{1}{3} < 1.\end{aligned}$$

38. Diverges by the nth-Term Test, since

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \neq 0.$$

39. Converges, because it is a geometric series with $r = -\frac{2}{3}$, so $|r| < 1$.

40. Diverges by the Ratio Test, since

$$\begin{aligned}\lim_{n\to\infty}\frac{a_{n+1}}{a_n} &= \lim_{n\to\infty}\frac{(n+1)!e^{-n-1}}{n!e^{-n}}\\ &= \lim_{n\to\infty}(n+1)e^{-1}\\ &= \infty.\end{aligned}$$

(The nth-Term Test can also be used.)

41. Diverges by the Ratio Test, since

$$\begin{aligned}\lim_{n\to\infty}\frac{a_{n+1}}{a_n} &= \lim_{n\to\infty}\frac{3^{n+1}}{(n+1)^3 2^{n+1}}\cdot\frac{n^3 2^n}{3^n}\\ &= \lim_{n\to\infty}\frac{3n^3}{(n+1)^3(2)}\\ &= \frac{3}{2} > 1.\end{aligned}$$

(The nth-Term Test can also be used.)

42. Converges by the Ratio Test, since

$$\begin{aligned}\lim_{n\to\infty}\frac{a_{n+1}}{a_n} &= \lim_{n\to\infty}\frac{(n+1)\ln(n+1)}{2^{n+1}}\cdot\frac{2^n}{n\ln n}\\ &= \lim_{n\to\infty}\frac{1}{2}\cdot\frac{n+1}{n}\cdot\frac{\ln(n+1)}{\ln n}\\ &= \lim_{n\to\infty}\frac{1}{2}\cdot\frac{1}{1}\cdot\frac{\frac{1}{(n+1)}}{1/n}\\ &\quad \text{(L'Hôpital's Rule)}\\ &= \frac{1}{2}\cdot\lim_{n\to\infty}\cdot\frac{n}{n+1}\\ &= \frac{1}{2} < 1\end{aligned}$$

43. Converges by the Ratio Test, since

$$\begin{aligned}\lim_{n\to\infty}\frac{a_{n+1}}{a_n} &= \lim_{n\to\infty}\frac{(n+1)!}{(2n+3)!}\cdot\frac{(2n+1)!}{n!}\\ &= \lim_{n\to\infty}\frac{n+1}{(2n+3)(2n+2)}\\ &= \lim_{n\to\infty}\frac{n+1}{4n^2+10n+2}\\ &= \lim_{n\to\infty}\frac{1}{8n+10} \qquad \text{(L'Hôpital's Rule)}\\ &= 0 < 1\end{aligned}$$

44. Converges by the Ratio Test, since

$$\begin{aligned}\lim_{n\to\infty}\frac{a_{n+1}}{a_n} &= \lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}\\ &= \lim_{n\to\infty}\frac{(n+1)n^n}{(n+1)^{n+1}}\\ &= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n\\ &= \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}\\ &= \frac{1}{e} < 1\end{aligned}$$

45. One possible answer: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (see Exploration 1 in this section) even though $\lim_{n\to\infty} \frac{1}{n} = 0.$

46. One possible answer:
Let $a_n = 2^{-n}$ and $b_n = 3^{-n}$
Then $\sum a_n$ and $\sum b_n$ are convergent geometric series, but $\sum \frac{a_n}{b_n} = \sum \left(\frac{3}{2}\right)^n$ is a divergent geometric series.

47. Almost, but the Ratio Test won't determine whether there is convergence or divergence at the endpoints of the interval.

48. $\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{4n-3} - \frac{1}{4n+1}\right)$

$s_1 = 1 - \frac{1}{5}$

$s_2 = \left(1 - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{9}\right) = 1 - \frac{1}{9}$

$s_3 = \left(1 - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{9}\right) + \left(\frac{1}{9} - \frac{1}{13}\right) = 1 - \frac{1}{13}$

$s_n = 1 - \frac{1}{4n+1}$

$S = \lim_{n\to\infty} s_n = 1$

49. $\sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)} = \sum_{n=1}^{\infty} \left(\frac{3}{2n-1} - \frac{3}{2n+1}\right)$

$s_1 = 3 - \frac{3}{3}$

$s_2 = (3-1) + \left(1 - \frac{3}{5}\right) = 3 - \frac{3}{5}$

$s_3 = (3-1) + \left(1 - \frac{3}{5}\right) + \left(\frac{3}{5} - \frac{3}{7}\right) = 3 - \frac{3}{7}$

$s_n = 3 - \frac{3}{2n+1}$

$S = \lim_{n\to\infty} s_n = 3$

50. $\frac{40n}{(2n-1)^2(2n+1)^2} = \frac{A}{(2n-1)^2} + \frac{B}{(2n+1)^2}$

$A(2n+1)^2 + B(2n-1)^2 = 40n$

$n = -\frac{1}{2} \Rightarrow 4B = -20 \Rightarrow B = -5$

$n = \frac{1}{2} \Rightarrow 4B = 20 \Rightarrow A = 5$

$\sum_{n=1}^{\infty} \frac{40n}{(2n-1)^2(2n+1)^2}$

$= \sum_{n=1}^{\infty} \left[\frac{5}{(2n-1)^2} - \frac{5}{(2n+1)^2}\right]$

$s_1 = 5 - \frac{5}{9}$

$s_2 = \left(5 - \frac{5}{9}\right) + \left(\frac{5}{9} - \frac{5}{25}\right) = 5 - \frac{5}{25}$

$s_3 = \left(5 - \frac{5}{9}\right) + \left(\frac{5}{9} - \frac{5}{25}\right) + \left(\frac{5}{25} - \frac{5}{49}\right) = 5 - \frac{5}{49}$

$s_n = 5 - \frac{5}{(2n+1)^2}$

$S = \lim_{n\to\infty} s_n = 5$

51. $\frac{2n+1}{n^2(n+1)^2} = \frac{A}{n^2} + \frac{B}{(n+1)^2}$

$A(n+1)^2 + Bn^2 = 2n+1$

$n = 0 \Rightarrow A = 1$

$n = -1 \Rightarrow B = -1$

$\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} = \sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right)$

$s_1 = 1 - \frac{1}{4}$

$s_2 = \left(1 - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{9}\right) = 1 - \frac{1}{9}$

$s_3 = \left(1 - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{9}\right) + \left(\frac{1}{9} - \frac{1}{16}\right) = 1 - \frac{1}{16}$

$s_n = 1 - \frac{1}{(n+1)^2}$

$S = \lim_{n\to\infty} s_n = 1$

52. $s_1 = 1 - \frac{1}{\sqrt{2}}$

$s_2 = \left(1 - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) = 1 - \frac{1}{\sqrt{3}}$

$s_3 = \left(1 - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}}\right)$

$= 1 - \frac{1}{\sqrt{4}}$

$s_n = 1 - \frac{1}{\sqrt{n+1}}$

$S = \lim_{n\to\infty} s_n = 1$

53. $s_1 = \frac{1}{\ln 3} - \frac{1}{\ln 2}$

$s_2 = \left(\frac{1}{\ln 3} - \frac{1}{\ln 2}\right) + \left(\frac{1}{\ln 4} - \frac{1}{\ln 3}\right) = \frac{1}{\ln 4} - \frac{1}{\ln 2}$

$s_3 = \left(\frac{1}{\ln 3} - \frac{1}{\ln 2}\right) + \left(\frac{1}{\ln 4} - \frac{1}{\ln 3}\right) + \left(\frac{1}{\ln 5} - \frac{1}{\ln 4}\right)$

$= \frac{1}{\ln 5} - \frac{1}{\ln 2}$

$s_n = \frac{1}{\ln(n+2)} - \frac{1}{\ln 2}$

$S = \lim_{n\to\infty} s_n = -\frac{1}{\ln 2}$

54. $s_1 = \tan^{-1} 1 - \tan^{-1} 2 = \frac{\pi}{4} - \tan^{-1} 2$

$s_2 = (\tan^{-1} 1 - \tan^{-1} 2) + (\tan^{-1} 2 - \tan^{-1} 3)$

$= \frac{\pi}{4} - \tan^{-1} 3$

$s_3 = (\tan^{-1} 1 - \tan^{-1} 2) + (\tan^{-1} 2 - \tan^{-1} 3) + (\tan^{-1} 3 - \tan^{-1} 4)$

$= \frac{\pi}{4} - \tan^{-1} 4$

$s_n = \frac{\pi}{4} - \tan^{-1}(n+1)$

$S = \lim_{n\to\infty} s_n = \frac{\pi}{4} - \lim_{n\to\infty} \tan^{-1} n = \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4}$

55. True; see Theorem 8.

56. False; the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ always converges at $x = a$. (The sum series is c_0.)

57. B; $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}}{(-3)^{n+1}}\cdot\frac{(-3)^n}{2^n}\right|$

$$= \lim_{n\to\infty}\frac{2}{3}$$
$$= \frac{2}{3}$$

58. C; $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|2x-3|^{n+1}}{n+1}\cdot\frac{n}{|2x-3|^n}$

$$= |2x-3|\cdot\lim_{n\to\infty}\frac{n}{n+1}$$
$$= |2x-3|$$

The series converges for $|2x-3|<1,$ which is equivalent to $\left|x-\frac{3}{2}\right|<\frac{1}{2},$ and diverges for $\left|x-\frac{3}{2}\right|>\frac{1}{2}.$ The radius of convergence is $\frac{1}{2}.$

59. E; consider first the series

$$\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{|\sin x|^n}{2^n n^2}$$

For any real number x, and for all n:

$$\frac{|\sin x|^n}{2^n n^2} \le \frac{1}{2^n n^2} < \frac{1}{2^n}$$

Since $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$ converges, $\sum_{n=1}^{\infty}\frac{|\sin x|^n}{2^n n^2}$ converges by the Direct Comparison Test.

Then, by Theorem 8, since $\sum_{n=1}^{\infty}\frac{(\sin x)^n}{2^n n^2}$ converges absolutely, it also converges.

60. D; $\sum_{n=1}^{\infty}\frac{3}{(3n-1)(3n+2)} = \sum_{n=1}^{\infty}\frac{1}{3n-1}-\frac{1}{3n+2}$

$$s_1 = \frac{1}{2}-\frac{1}{5}$$
$$s_2 = \left(\frac{1}{2}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{8}\right) = \frac{1}{2}-\frac{1}{8}$$
$$s_3 = \left(\frac{1}{2}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{8}\right)+\left(\frac{1}{8}-\frac{1}{11}\right) = \frac{1}{2}-\frac{1}{11}$$
$$s_n = \frac{1}{2}-\frac{1}{3n+2}$$
$$S = \lim_{n\to\infty}\left(\frac{1}{2}-\frac{1}{3n+2}\right) = \frac{1}{2}$$

61. **(a)** For $k\le N,$ it's obvious that

$$S_k = a_1+\cdots+a_k \le a_1+\cdots+a_N + \sum_{n=N+1}^{\infty}c_n.$$

For all $k>N,$

$$\begin{aligned} S_k &= a_1+\cdots+a_k \\ &= a_1+\cdots+a_N+a_{N+1}+\cdots+a_k \\ &\le a_1+\cdots+a_N+c_{N+1}+\cdots+c_k \\ &\le a_1+\cdots+a_N+\sum_{n=N+1}^{\infty}c_n \end{aligned}$$

(b) Since all of the a_n are nonnegative, the partial sums of the series form a nondecreasing sequence of real numbers. Part (a) shows that this sequence is bounded above, so it must converge to a limit.

62. **(a)** For $k\le N,$ it's obvious that

$$d_1+\cdots+d_k \le d_1+\cdots+d_N+\sum_{n=N+1}^{\infty}a_n.$$

For all $k>N,$

$$\begin{aligned} &d_1+\cdots+d_k \\ &= d_1+\cdots+d_N+d_{N+1}+\cdots+d_k \\ &\le d_1+\cdots+d_N+a_{N+1}+\cdots+a_k \\ &\le d_1+\cdots+d_N+\sum_{n=N+1}^{\infty}a_n \end{aligned}$$

(b) If $\sum a_n$ converged, part (a) would imply that $\sum d_n$ was also convergent.

63. Answers will vary.

64. $\frac{1}{1-x} = \sum_{n=0}^{\infty}x^n$

Differentiate:

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}nx^{n-1}$$

Multiply by x:

$$\frac{x}{(1-x)^2} = \sum_{n=0}^{\infty}nx^n$$

Differentiate:

$$\frac{d}{dx}\left(\frac{x}{(1-x)^2}\right)=\frac{(1-x)^2(1)-(x)(2)(1-x)(-1)}{(1-x)^4}$$
$$=\frac{(1-x)+2x}{(1-x)^3}$$
$$=\frac{x+1}{(1-x)^3}$$
$$\frac{x+1}{(1-x)^3}=\sum_{n=0}^{\infty} n^2x^{n-1}$$

Multiply by x:

$$\frac{x(x+1)}{(1-x)^3}=\sum_{n=0}^{\infty} n^2x^n$$

Let $x=\frac{1}{2}$:

$$\frac{\frac{1}{2}\left(\frac{3}{2}\right)}{\left(\frac{1}{2}\right)^3}=\sum_{n=0}^{\infty} n^2\left(\frac{1}{2}\right)^n$$
$$6=\sum_{n=0}^{\infty}\frac{n^2}{2^n}$$

The sum is 6.

Section 10.5 Testing Convergence at Endpoints (pp. 517–530)

Exploration 1 The *p*-Series Test

1. We first note that the Integral Test applies to any series of the form $\sum\frac{1}{n^p}$ where p is positive. This is because the function $f(x)=x^{-p}$ is continuous and positive for all $x>0$, and $f'(x)=-p\bullet x^{-p-1}$ is negative for all $x>0$.
   If $p>1$:
   $$\int_1^{\infty}\frac{1}{x^p}dx=\lim_{k\to\infty}\int_1^k\frac{1}{x^p}dx=\lim_{k\to\infty}\left(\left.\frac{x^{-p+1}}{-p+1}\right]_1^k\right)$$
   $$=\lim_{k\to\infty}\left(\frac{1}{1-p}\cdot\left(\frac{1}{k^{p-1}}-1\right)\right)$$
   $$=0+\frac{1}{p-1}\quad(\text{since } p-1>0)$$
   $$=\frac{1}{p-1}<\infty.$$
   The series converges by the Integral Test.

2. If $0<p<1$:
   $$\int_1^{\infty}\frac{1}{x^p}dx=\lim_{k\to\infty}\int_1^k\frac{1}{x^p}dx$$
   $$=\lim_{k\to\infty}\left(\left.\frac{x^{-p+1}}{-p+1}\right]_1^k\right)$$
   $$=\lim_{k\to\infty}\left(\frac{1}{1-p}\cdot(k^{1-p}-1)\right)$$
   $$=\infty\quad(\text{since } 1-p>0).$$
   The series diverges by the Integral Test.
   If $p\le 0$, the series diverges by the nth-Term Test. This completes the proof for $p<1$.

3. If $p=1$:
   $$\int_1^{\infty}\frac{1}{x^p}dx=\lim_{k\to\infty}\int_1^k\frac{1}{x}dx$$
   $$=\lim_{k\to\infty}\left(\ln x\Big]_1^k\right)$$
   $$=\lim_{k\to\infty}\ln k$$
   $$=\infty.$$
   The series diverges by the Integral Test.

Exploration 2 The Maclaurin Series of a Strange Function

1. Since $f^{(n)}(0)=0$ for all n, the Maclaurin Series for f has all zero coefficients! The series is simply $\sum_{n=0}^{\infty}0\cdot x^n=0$.

2. The series converges (to 0) for all values of x.

3. Since $f(x)=0$ only at $x=0$, the only place that this series actually converges to $f(x)$ is at $x=0$.

Quick Review 10.5

1. Converges, since it is of the form $\int_1^{\infty}\frac{1}{x^p}dx$ with $p>1$.

2. Diverges, limit comparison test with integral of $\frac{1}{x}$.

3. Diverges, comparison test with integral of $\frac{1}{x}$.

4. Converges, comparison test with integral of $\frac{2}{x^2}$.

5. Diverges, limit comparison test with integral of $\frac{1}{\sqrt{x}}$.

6. Yes; $f(x) = \frac{3}{x} > 0$ for $x > 0$.

$f'(x) = -\frac{3}{x^2} < 0$ for $x \neq 0$.

Therefore f is positive and decreasing on $(0, \infty)$.

7. Yes; $f(x) = \frac{7x}{x^2 - 8} > 0$ when $x > 0$ and $x^2 > 8$, or when $x < 0$ and $x^2 < 8$.

$f'(x) = -\frac{7(x^2+8)}{(x^2-8)^2} < 0$ for $x^2 \neq 8$.

Therefore, f is positive and decreasing on $\left(2\sqrt{2}, \infty\right)$. (Also on $\left(-2\sqrt{2}, 0\right)$, but that is not the kind of interval we are looking for.)

8. No; $f(x) = \frac{3+x^2}{3-x^2} > 0$ only when $3 - x^2 > 0$, so $f(x)$ is positive only for $-\sqrt{3} < x < \sqrt{3}$.

9. No; on any interval (N, ∞), will oscillate between -1 and 1, so $f(x)$ will oscillate between positive and negative values, as well.

10. No; $x > 1$ implies $\frac{1}{x} < 1$, so $f(x) = \ln\left(\frac{1}{x}\right) < 0$ for all $x > 1$.

Section 10.5 Exercises

1.
$$\int_1^{\infty} \frac{1}{\sqrt[3]{x}}\,dx = \lim_{k\to\infty} \int_1^k x^{-1/3}\,dx$$
$$= \lim_{k\to\infty} \left[\frac{3}{2}x^{2/3}\right]_1^k$$
$$= \lim_{k\to\infty} \left(\frac{3}{2}k^{2/3} - \frac{3}{2}(1)\right)$$
$$= \infty$$

Since $\int_1^{\infty} \frac{1}{\sqrt[3]{x}}\,dx$ diverges, it follows from the Integral Test that $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$ also diverges.

2.
$$\int_1^{\infty} x^{-3/2}\,dx = \lim_{k\to\infty} \int_1^k x^{-3/2}\,dx$$
$$= \lim_{k\to\infty} \left[-\frac{1}{2}x^{-1/2}\right]_1^k$$
$$= \lim_{k\to\infty} \left(-\frac{1}{2\sqrt{k}} + \frac{1}{2\sqrt{1}}\right)$$
$$= 0 + \frac{1}{2}$$
$$= \frac{1}{2}$$

Since $\int_1^{\infty} x^{-3/2}\,dx$ converges, it follows from the Integral Test that $\sum_{n=1}^{\infty} n^{-3/2}$ also converges.

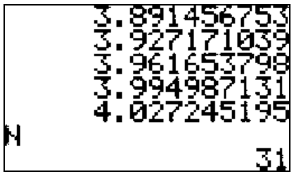
3.
$$S_1 = 1$$
$$S_2 = 1 + \frac{1}{2} = \frac{3}{2}$$
$$S_3 = \frac{3}{2} + \frac{1}{3} = \frac{11}{6}$$
$$S_4 = \frac{11}{6} + \frac{1}{4} = \frac{50}{24} = \frac{25}{12}$$
$$S_5 = \frac{25}{12} + \frac{1}{5} = \frac{137}{60}$$
$$S_6 = \frac{137}{60} + \frac{1}{6} = \frac{147}{60} = \frac{49}{20}$$

4. The quickest way is to use a calculator.

```
1→N:1→T
                 1
N+1→N:T+1/N→T
               1.5
       1.833333333
       2.083333333
       2.283333333
```

Notice that T contains S_N.

Keep pressing ENTER until T > 4.



31 is the first value of N for which $S_N > 4$.
Therefore, $k = 31$.

5. For n large, $\frac{3n-1}{n^2+1}$ behaves like $\frac{3}{n}$.

Let $a_n = \frac{3n-1}{n^2+1}$ and $b_n = \frac{1}{n}$.
Then $a_n > 0$ and $b_n > 0$ for $n > 0$, and

$$\begin{aligned}
c &= \lim_{n\to\infty} \frac{a_n}{b_n} \\
&= \lim_{n\to\infty} \frac{\frac{(3n-1)}{(n^2+1)}}{\frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{n(3n-1)}{n^2+1} \\
&= \lim_{n\to\infty} \frac{3n^2-n}{n^2+1} \\
&= \lim_{n\to\infty} \frac{3-\frac{1}{n}}{1+\frac{1}{n^2}} \\
&= 3
\end{aligned}$$

Since $0 < c < \infty$, and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (by the p-Test), $\sum_{n=1}^{\infty} \frac{3n-1}{n^2+1}$ diverges by the Limit Comparison Test.

6. For n large, $\frac{2^n}{3^n+1}$ behaves like $\frac{2^n}{3^n}$.

Let $a_n = \frac{2^n}{3^n+1}$ and $b_n = \frac{2^n}{3^n}$.
Then $a_n > 0$ and $b_n > 0$ for $n \ge 0$, and

$$\begin{aligned}
c &= \lim_{n\to\infty} \frac{a_n}{b_n} \\
&= \lim_{n\to\infty} \frac{\left[\frac{(2^n)}{((3^n)+1)}\right]}{\left(\frac{2}{3}\right)^n} \\
&= \lim_{n\to\infty} \frac{2^n}{3^n+1} \cdot \frac{3^n}{2^n} \\
&= \lim_{n\to\infty} \frac{3^n}{3^n+1} \\
&= \lim_{n\to\infty} \frac{1}{1+\frac{1}{3^n}} \\
&= 1
\end{aligned}$$

Since $0 < c < \infty$, and $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ converges $\left(\text{geometric series with } r = \frac{2}{3}\right)$, $\sum_{n=1}^{\infty} \frac{2^n}{3^n+1}$ converges by the Limit Comparison Test.

7. Diverges
Method 1: Use the Integral Test with
$\int_1^{\infty} \frac{5}{x+1}\,dx.$
Method 2: Use the Limit Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{n}$ (which diverges by the p-test).

8. Diverges
Method 1: Rewrite the series as $3\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ and use the p-test.
Method 2: Use the Integral Test with
$\int_1^{\infty} \frac{3}{\sqrt{x}}\,dx.$
Method 3: Use the Direct Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ (which diverges by the p-test).
Method 4: Use the Limit Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ (which diverges by the p-test).

9. Diverges
Method 1: Use the Integral Test with
$\int_1^{\infty} \frac{\ln x}{x}\,dx.$
Method 2: Since $\ln n > 1$ for $n > 3$, use the

Direct Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{n}$ (which diverges by the p-test).
Method 3: Use the Limit Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{n}$ (which diverges by the p-test).

10. Diverges
Method 1: Use the Integral Test with
$\int_1^{\infty} \frac{1}{2x-1}\,dx.$
Method 2: Use the Limit Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{n}$ (which diverges by the p-test).
Method 3: Use the Direct Comparison Test:
$$2n-1 < 2n \Rightarrow \frac{1}{2n-1} > \frac{1}{2n} = \frac{1}{2}\cdot\frac{1}{n}.$$
Compare the given series with $\frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$ (which diverges by the p-test).

11. Diverges; geometric series with
$r = \frac{1}{\ln 2} \approx 1.44.$

12. Converges; geometric series with
$r = \frac{1}{\ln 3} \approx 0.91.$

13. Diverges by the nth-Term Test:
$$\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{x\to 0} \frac{\sin x}{x} = 1 \ne 0$$

14. Converges
Method 1: Use the Integral Test:
$$\int_0^{\infty} \frac{e^x}{1+e^{2x}}\,dx = \lim_{k\to\infty} \tan^{-1}(e^x)\Big|_0^k = \lim_{k\to\infty} \tan^{-1}(e^k) - \tan^{-1}(1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$
Method 2: Use the Direct Comparison Test:
$$\frac{e^n}{1+e^{2n}} < \frac{e^n}{e^{2n}} = \left(\frac{1}{e}\right)^n$$
Compare the given series with $\sum_{n=0}^{\infty}\left(\frac{1}{e}\right)^n$ $\left(\text{geometric series with } r = \frac{1}{e} \approx 0.37\right)$.
Method 3: Use the Limit Comparison Test with $\sum_{n=0}^{\infty}\left(\frac{1}{e}\right)^n$ (geometric series with $\left.r = \frac{1}{e} \approx 0.37\right)$.

15. Converges
Method 1: Use the Direct Comparison Test:
$$\frac{\sqrt{n}}{n^2+1} < \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$$
Compare the given series with $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ (which converges by the p-test).
Method 2: Use the Limit Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ (which converges by the p-test).

16. Converges; use the Limit Comparison Test.
Let $a_n = \frac{5n^3-3n}{n^2(n+2)(n^2+5)}$ and $b_n = \frac{1}{n^2}$
Then $a_n > 0$ and $b_n > 0$ for $n \ge 1$, and
$$c = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{5n^5-3n^3}{n^5+2n^4+5n^3+10n^2} = 5$$
Since $0 < c < \infty$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-Test, $\sum_{n=1}^{\infty} \frac{5n^3-3n}{n^2(n+2)(n^2+5)}$ also converges.

17. Diverges; use the nth-Term Test:
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3^{n-1}+1}{3^n} = \lim_{n\to\infty} \frac{1+\frac{1}{3^{n-1}}}{3} = \frac{1}{3} \ne 0$$
Since the sequence of terms does not converge to 0, the series diverges.

18. First, check if the series converges absolutely. Use the Limit Comparison Test with $\sum_{n=2}^{\infty}\frac{1}{n}$ to show that $\sum_{n=2}^{\infty}\frac{1}{\ln n}$ diverges. So, $\sum_{n=2}^{\infty}(-1)^n\frac{1}{\ln n}$ does not converge absolutely.
Next, use the Alternating Series Test.

(1) $u_n = \frac{1}{\ln n} > 0$ for $n \ge 2$.

(2) We know that $\ln x$ is an increasing function, so
$n+1 > n \Rightarrow \ln(n+1) > \ln n$
$\Rightarrow \frac{1}{\ln(n+1)} < \frac{1}{\ln n}$. Thus, the u_n are decreasing.

(3) $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{\ln n} = 0$

Therefore, $\sum_{n=2}^{\infty}(-1)^n\frac{1}{\ln n}$ converges.

19. Diverges by the nth-Term Test. Use L'Hôpital's Rule ten times to show that:
$\lim_{x\to\infty}\frac{10^x}{x^{10}} = \lim_{x\to\infty}\frac{(\ln 10)^{10}10^x}{10!} = \infty$
This implies that
$\lim_{n\to\infty}|a_n| \ne 0$, so $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{10^n}{n^{10}}$ diverges.

20. Converges by the Alternating Series Test. If $u_n = \frac{\sqrt{n}+1}{n+1}$, then $\{u_n\}$ is a decreasing sequence of positive terms with $\lim_{n\to\infty} u_n = 0$.
(To show that u_n is decreasing, let
$f(x) = \frac{\sqrt{x}+1}{x+1}$ and observe that
$$f'(x) = \frac{(x+1)\left(\frac{1}{2\sqrt{x}}\right) - \left(\sqrt{x}+1\right)(1)}{(x+1)^2}$$
$$= \frac{1-x-2\sqrt{x}}{2(x+1)^2\sqrt{x}},$$
which is negative, at least for $x \ge 1$.)

21. Diverges by the nth-Term Test, since $\frac{\ln n}{\ln n^2} = \frac{\ln n}{2\ln n} = \frac{1}{2}$, which means each term is $\pm\frac{1}{2}$.

22. Diverges by the Limit Comparison Test:
Let $a_n = \frac{1}{n} - \frac{1}{n^2}$ and $b_n = \frac{1}{n}$.
Then $a_n > 0$ and $b_n > 0$ for $n \ge 2$ and
$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{1}{n}-\frac{1}{n^2}}{\frac{1}{n}}$$
$$= \lim_{n\to\infty}\frac{n-1}{n}$$
$$= \lim_{n\to\infty}\left(1-\frac{1}{n}\right)$$
$$= 1.$$
Since $\sum_{n=1}^{\infty} b_n$ diverges, $\sum_{n=1}^{\infty} a_n$ also diverges.

23. Converges conditionally. First, check for absolute convergence:
$\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\left(\frac{1}{n}+\frac{1}{n^2}\right)$ diverges by the Direct Comparison Test, since $\frac{1}{n}+\frac{1}{n^2} > \frac{1}{n}$ and $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges (by the p-Test).
Therefore, $\sum_{n=1}^{\infty}(-1)^n\frac{n+1}{n^2}$ does not converge absolutely.
Next, use the Alternating Series Test to check for conditional convergence:

(1) $u_n = \frac{1}{n}+\frac{1}{n^2} > 0$ Clear

(2) $\frac{d}{dx}\left(\frac{1}{x}+\frac{1}{x^2}\right) = -\frac{1}{x^2} - \frac{2}{x^3} < 0$, for $x > 0$.
Thus, the u_n are decreasing.

(3) $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n^2}\right) = 0$

Therefore, $\sum_{n=1}^{\infty}(-1)^n\frac{n+1}{n^2}$ converges.
Truncation error after 99 terms $\le |u_{100}| = 0.0101$

24. Converges absolutely.

$\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}(0.1)^n$ is a geometric series with $r = 0.1$. Truncation error after 99 terms $\le |u_{100}| = 10^{-100}$

25. Converges conditionally. First, check for absolute convergence:

$\sum_{n=2}^{\infty}|a_n| = \sum_{n=2}^{\infty}\frac{1}{n\ln n}$ diverges by the Integral Test, since $\int_2^{\infty}\frac{1}{x\ln x}dx = \lim_{k\to\infty}\left[\ln|\ln x|\right]_2^k = \infty$

Therefore, $\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{n\ln n}$ does not converge absolutely.

Next, use the Alternating Series Test to check for conditional convergence:

(1) $u_n = \frac{1}{n\ln n} > 0$ for $n \ge 2$.

(2) $\ln x$ is everywhere increasing, so

$$\begin{aligned} n+1 > n &\Rightarrow \ln(n+1) > \ln n \\ &\Rightarrow (n+1)\ln(n+1) > n\ln n \\ &\Rightarrow \frac{1}{(n+1)\ln(n+1)} < \frac{1}{n\ln n} \end{aligned}$$

Thus, the u_n are decreasing.

(3) $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{n\ln n} = 0$

Therefore, $\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{n\ln n}$ converges.

Truncation error after 99 terms $\le |u_{100}| = 0.00217$

26. Converges absolutely

$\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty} n^2\left(\frac{2}{3}\right)^n$ converges by the Ratio Test, since

$$\begin{aligned} \lim_{n\to\infty}\frac{(n+1)^2\left(\frac{2}{3}\right)^{n+1}}{n^2\left(\frac{2}{3}\right)^n} &= \frac{2}{3}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 \\ &= \frac{2}{3} < 1. \end{aligned}$$

Truncation error after 99 terms $\le u_{100} = 2.46\times10^{-14}$

27. Diverges by the nth-Term Test:

$$\begin{aligned} \lim_{n\to\infty}|a_n| &= \lim_{n\to\infty}\frac{n!}{2^n} \\ &= \lim_{n\to\infty}\left(\frac{n}{2}\cdot\frac{n-1}{2}\cdot\frac{n-2}{2}\cdots\frac{3}{2}\cdot\frac{2}{2}\cdot\frac{1}{2}\right) \\ &\ge \lim_{n\to\infty}\left(\frac{n}{2}\cdot1\cdot1\cdots1\cdot1\cdot\frac{1}{2}\right) \\ &= \lim_{n\to\infty}\frac{n}{4} \\ &= \infty \end{aligned}$$

Since the terms do not converge to 0, $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n!}{2^n}$ diverges.

28. Converges absolutely

Since $|a_n| = \frac{|\sin n|}{n^2} \le \frac{1}{n^2}$, use the Direct Comparison Test with $\sum_{n=1}^{\infty}\frac{1}{n^2}$ (which converges by the p-test).

29. Converges conditionally; first, check for absolute convergence:

$\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{1}{1+\sqrt{n}}$ diverges by the Limit Comparison Test (use $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$, which diverges by the p-Test).

Therefore, $\sum_{n=1}^{\infty}\frac{(-1)^n}{1+\sqrt{n}}$ does not converge absolutely.

Next, use the Alternating Series Test to check for conditional convergence:

(1) $u_n = \frac{1}{1+\sqrt{n}} > 0$ Clear

(2)

$$\begin{aligned} n+1 > n &\Rightarrow \sqrt{n+1} > \sqrt{n} \\ &\Rightarrow 1+\sqrt{n+1} > 1+\sqrt{n} \\ &\Rightarrow \frac{1}{1+\sqrt{n+1}} < \frac{1}{1+\sqrt{n}} \end{aligned}$$

Thus the u_n are decreasing.

(3) $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{1+\sqrt{n}} = 0$

Therefore, $\sum_{n=1}^{\infty}\frac{(-1)^n}{1+\sqrt{n}}$ converges.

30. Converges absolutely

$\sum_{n=1}^{\infty}\left|\frac{\cos n\pi}{n\sqrt{n}}\right| = \sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n\sqrt{n}}\right| = \sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$, which converges by the p-Test.

31. Converges conditionally

$\sum_{n=1}^{\infty}\left|\frac{\cos n\pi}{n}\right| = \sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$ (which diverges by the p-Test).

Therefore, $\sum_{n=1}^{\infty}\frac{\cos n\pi}{n}$ does not converge absolutely.

Next, use the Alternating Series Test to show that $\sum_{n=1}^{\infty}\frac{\cos n\pi}{n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges. (See Example 4).

32. Converges conditionally; first, check for absolute convergence:

$\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+\sqrt{n+1}}$ diverges by the Limit Comparison Test. Let $a_n = \frac{1}{\sqrt{n}+\sqrt{n+1}}$ and $b_n = \frac{1}{\sqrt{n}}$. Then

$$\begin{aligned} c &= \lim_{n\to\infty}\frac{a_n}{b_n} \\ &= \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n}+\sqrt{n+1}} \\ &= \lim_{n\to\infty}\frac{1}{1+\sqrt{1+\frac{1}{n}}} \\ &= \frac{1}{2} \end{aligned}$$

Since $0 < c < \infty$ and $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges (by the p-Test), $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+\sqrt{n+1}}$ also diverges.

Therefore, $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}+\sqrt{n+1}}$ does not converge absolutely.

Next, use the Alternating Series Test to check for conditional convergence:

(1) $u_n = \frac{1}{\sqrt{n}+\sqrt{n+1}} > 0$ Clear

(2) $\sqrt{n+1}+\sqrt{n+2} > \sqrt{n}+\sqrt{n+1}$

$\Rightarrow \frac{1}{\sqrt{n+1}+\sqrt{n+2}} < \frac{1}{\sqrt{n}+\sqrt{n+1}}$

Thus, the u_n are decreasing.

(3) $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{\sqrt{n}+\sqrt{n+1}} = 0$

Therefore, $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}+\sqrt{n+1}}$ converges.

33. The positive terms

$2+\frac{4}{3^2}+\frac{6}{5^2}+\cdots+\frac{2n+2}{(2n+1)^2}+\cdots$ diverge to ∞

and the negative terms

$-\frac{3}{2^2}-\frac{5}{4^2}-\frac{7}{6^2}-\cdots-\frac{2n+1}{(2n)^2}-\cdots$ diverge to $-\infty$. Answers will vary. Here is one possibility.

(a) Add positive terms until the partial sum is greater than 2. Then add negative terms until the partial sum is less than –2. Then add positive terms until the partial sum is greater than 4. Then add negative terms until the partial sum is less than –4. Repeat this process so that the partial sums swing arbitrarily far in both directions.

(b) Add positive terms until the partial sum is greater than 4. Then add negative terms until the partial sum is less than 4. Continue in this manner indefinitely, always closing in on 4.

34. The positive terms

$$\frac{1}{3\ln 3}+\frac{1}{5\ln 5}+\frac{1}{7\ln 7}+\cdots+\frac{1}{(2n+1)\ln(2n+1)}+\cdots$$

diverge to ∞ and the negative terms

$-\frac{1}{2\ln 2}-\frac{1}{4\ln 4}-\frac{1}{6\ln 6}-\cdots-\frac{1}{(2n)\ln(2n)}-\cdots$

diverge to $-\infty$. Answers will vary. Here is one possibility.

(a) Add positive terms until the partial sum is greater than 1. Then add negative terms until the partial sum is less than –1. Then add positive terms until the partial sum is greater than 2. Then add negative terms until the partial sum is less than –2.

Repeat this process so that the partial sums swing arbitrarily far in both directions.

(b) Add positive terms until the partial sum is greater than 4. Then add negative terms until the partial sum is less than 4. Continue in this manner indefinitely, always closing in on 4.

35. $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}|x|^n$, geometric series with $r = |x|$. The series converges absolutely for $|x| < 1$, diverges for $|x| \geq 1$.

(a) Interval of convergence: $(-1, 1)$

(b) Series converges absolutely on $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$

(c) None

36. $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}|x+5|^n$, geometric series with $r = |x+5|$. This series converges absolutely for $|x+5| < 1$, diverges for $|x+5| \geq 1$.

(a) Interval of convergence: $(–6, –4)$

(b) Series converges absolutely on $(–6, –4)$

(c) None

37. $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}|4x+1|^n$, geometric series with $r = |4x+1|$. The series converges absolutely for $|4x+1| < 1$, diverges for $|4x+1| \geq 1$.

(a) Interval of convergence: $\left(-\frac{1}{2}, 0\right)$

(b) Series converges absolutely on $\left(-\frac{1}{2}, 0\right)$.

(c) None

38. $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{|3x-2|^n}{n}$; use the Ratio Test.

$$\lim_{n\to\infty}\frac{|3x-2|^{n+1}}{n+1}\cdot\frac{n}{|3x-2|^n}$$
$$= |3x-2| \cdot \lim_{n\to\infty}\frac{n}{n+1}$$
$$= |3x-2|$$

The series converges absolutely for $|3x-2| < 1$, or $x \in \left(\frac{1}{3}, 1\right)$;

the series diverges for $|3x-2| > 1$

Check $x = 1$: $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges by the p-Test.

Check $x = \frac{1}{3}$: $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ does not converge absolutely, but it converges conditionally by the Alternating Series Test.

(a) Interval of convergence: $\left[\frac{1}{3}, 1\right)$

(b) Series converges absolutely on $\left(\frac{1}{3}, 1\right)$.

(c) Series converges conditionally at $x = \frac{1}{3}$.

39. $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\left|\frac{x-2}{10}\right|^n$, geometric series with $r = \left|\frac{x-2}{10}\right|$.

The series converges absolutely for $\left|\frac{x-2}{10}\right| < 1$, diverges for $\left|\frac{x-2}{10}\right| \geq 1$.

(a) Interval of convergence: $(–8, 12)$

(b) Series converges absolutely on $(–8, 12)$.

(c) None

40. $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{n|x|^n}{n+2}$; use the Ratio Test.

$$\lim_{n\to\infty}\frac{(n+1)|x|^{n+1}}{n+3}\cdot\frac{n+2}{n|x|^n}$$
$$=|x|\lim_{n\to\infty}\frac{n^2+3n+2}{n^2+3n}$$
$$=|x|$$

The series converges absolutely for $|x|<1$, diverges for $|x|>1$. At $x=\pm 1$, the series diverges by the nth-Term Test.

(a) Interval of convergence: (–1, 1)

(b) Series converges absolutely on (–1, 1).

(c) None

41. $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{|x|^n}{n\sqrt{n}3^n}$; use the Ratio Test.

$$\lim_{n\to\infty}\frac{|x|^{n+1}}{(n+1)\sqrt{n+1}\cdot 3^{n+1}}\bullet\frac{n\sqrt{n}\cdot 3^n}{|x|^n}$$
$$=\frac{|x|}{3}\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{3/2}$$
$$=\frac{|x|}{3}$$

The series converges absolutely for $|x|<3$, diverges for $|x|>3$. When $|x|=3$, the series also converges absolutely because $\sum_{n=1}^{\infty}\frac{|x|^n}{n\sqrt{n}3^n}=\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$, which converges by the p-Test.

(a) Interval of convergence: [–3, 3]

(b) Series converges absolutely on [–3, 3].

(c) None

42. $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{|x|^{2n+1}}{n!}$; use the Ratio Test.

$$\lim_{n\to\infty}\frac{|x|^{2n+3}}{(n+1)!}\cdot\frac{n!}{|x|^{2n+1}}=x^2\lim_{n\to\infty}\frac{1}{n+1}=0$$

The series converges absolutely for all real numbers.

(a) Interval of convergence: $(-\infty, \infty)$

(b) Series converges absolutely for all real numbers

(c) None

43. $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{n|x+3|^n}{5^n}$; use the Ratio Test.

$$\lim_{n\to\infty}\frac{(n+1)|x+3|^{n+1}}{5^{n+1}}\cdot\frac{5^n}{n|x+3|^n}$$
$$=\frac{|x+3|}{5}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)$$
$$=\frac{|x+3|}{5}$$

The series converges absolutely for $|x+3|<5$ and diverges for $|x+3|>5$. When $|x+3|=5$, the series diverges by the nth-Term Test.

(a) Interval of convergence: (–8, 2)

(b) Series converges absolutely on (–8, 2).

(c) None

44. $\sum_{n=1}^{\infty}|a_n| = \frac{n|x|^n}{4^n(n^2+1)}$; use the Ratio Test.

$$\lim_{n\to\infty}\frac{(n+1)|x|^{n+1}}{4^{n+1}[(n+1)^2+1]}\cdot\frac{4^n(n^2+1)}{n|x|^n}$$
$$=\frac{|x|}{4}\lim_{n\to\infty}\frac{n^3+n^2+n+1}{n^3+2n^2+2n}$$
$$=\frac{|x|}{4}$$

The series converges absolutely for $|x|<4$ and diverges for $|x|>4$.

Check $x=4$: $\sum_{n=1}^{\infty}\frac{n}{n^2+1}$ diverges by the Limit Comparison Test (use $\sum_{n=1}^{\infty}\frac{1}{n}$, which diverges by the p-Test).

Check $x=-4$: $\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^2+1}$ does not converge absolutely, but it converges conditionally by the Alternating Series Test.

(a) Interval of convergence: [–4, 4)

(b) Series converges absolutely on (–4, 4).

(c) Series converges conditionally at $x = -4$.

45. $\sum_{n=1}^{\infty}|a_n| = \frac{\sqrt{n}|x|^n}{3^n}$; use the Ratio Test.

$$\lim_{n\to\infty}\frac{\sqrt{n+1}|x|^{n+1}}{3^{n+1}}\cdot\frac{3^n}{\sqrt{n}|x|^n}$$
$$=\frac{|x|}{3}\lim_{n\to\infty}\sqrt{1+\frac{1}{n}}$$
$$=\frac{|x|}{3}$$

The series converges absolutely for $|x| < 3$ and diverges for $|x| > 3$. When $|x| = 3$, the series diverges by the nth-Term Test.

(a) Interval of convergence: (–3, 3)

(b) Series converges absolutely on (–3, 3).

(c) None

46. $\sum_{n=1}^{\infty}|a_n| = n!|x-4|^n$; use the Ratio Test.

$$\lim_{n\to\infty}\frac{(n+1)!|x-4|^{n+1}}{n!|x-4|^n} = \lim_{n\to\infty}(n+1)|x-4|$$
$$=\begin{cases}0 & x=4\\ \infty & x\neq 4\end{cases}$$

(a) Series only converges at $x = 4$.

(b) Series converges absolutely at $x = 4$.

(c) None

47. $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}2^n(n+1)|x-1|^n$; use the Ratio Test.

$$\lim_{n\to\infty}\frac{2^{n+1}(n+2)|x-1|^{n+1}}{2^n(n+1)|x-1|^n}$$
$$=2|x-1|\lim_{n\to\infty}\left(\frac{n+2}{n+1}\right)$$
$$=2|x-1|$$

The series converges absolutely for $|x-1| < \frac{1}{2}$ and diverges for $|x-1| > \frac{1}{2}$. When $|x-1| = \frac{1}{2}$, the series diverges by the nth-Term Test.

(a) Interval of convergence: $\left(\frac{1}{2}, \frac{3}{2}\right)$

(b) Series converges absolutely on $\left(\frac{1}{2}, \frac{3}{2}\right)$.

(c) None

48. $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{|4x-5|^{2n+1}}{n^{3/2}}$; use the Ratio Test.

$$\lim_{n\to\infty}\frac{|4x-5|^{2n+3}}{(n+1)^{3/2}}\cdot\frac{n^{3/2}}{|4x-5|^{2n+1}}$$
$$=|4x-5|^2\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{3/2}$$
$$=|4x-5|^2$$

The series converges absolutely for $|4x-5| < 1$, or $x \in \left(1, \frac{3}{2}\right)$; the series diverges for $|4x-5| > 1$.

Check $x = 1$: $\sum_{n=1}^{\infty}\frac{(-1)^{2n+1}}{n^{3/2}} = -\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ converges absolutely by the p-Test.

Check $x = \frac{3}{2}$: $\sum_{n=1}^{\infty}\frac{1^{2n+1}}{n^{3/2}} = \sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ converges absolutely by the p-Test.

(a) Interval of convergence: $\left[1, \frac{3}{2}\right]$

(b) Series converges absolutely on $\left[1, \frac{3}{2}\right]$

(c) None

49. $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{|x+\pi|^n}{\sqrt{n}}$; use the Ratio Test.

$$\lim_{n\to\infty}\frac{|x+\pi|^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{|x+\pi|^n}$$
$$=|x+\pi|\lim_{n\to\infty}\sqrt{\frac{n}{n+1}}$$
$$=|x+\pi|$$

The series converges absolutely for $|x+\pi|<1$, or $x \in (-\pi-1, -\pi+1)$; the series diverges for $|x+\pi|>1$.

Check $x=-\pi+1$: $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by the p-Test with $p=\frac{1}{2}$.

Check $x=-\pi-1$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ does not converge absolutely, but it converges conditionally by the Alternating Series Test.

(a) Interval of convergence: $[-\pi-1, -\pi+1)$

(b) Series converges absolutely on $(-\pi-1, -\pi+1)$.

(c) Series converges conditionally at $x=-\pi-1$.

50. $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} |\ln x|^n$, geometric series with $r=|\ln x|$.

The series converges absolutely for $|\ln x|<1$, or for $e^{-1}<x<e^{1}$; the series diverges for $|\ln x| \ge 1$.

(a) Interval of convergence: $\left(\frac{1}{e}, e\right)$

(b) Series converges absolutely on $\left(\frac{1}{e}, e\right)$.

(c) None

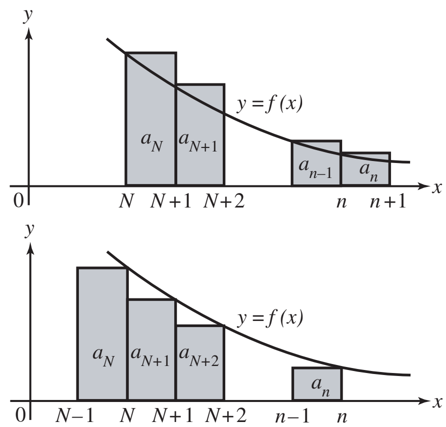
51. $n = 13\times10^9 \cdot 365 \cdot 24 \cdot 3600 = 4.09968\times10^{17}$
$\ln(n+1) < \text{sum} < 1+\ln n$
$\ln(4.09968\times10^{17}+1) < \text{sum} < 1+\ln(4.09968\times10^{17})$
$40.5548\ldots < \text{sum} < 41.5548\ldots$
$40.554 < \text{sum} < 41.555$

52. Comparing areas in the figures, we have for all $n \ge 1$,

$$\int_1^{n+1} f(x)\,dx < a_1 + \cdots + a_n < a_1 + \int_1^n f(x)\,dx.$$

If the integral diverges, it must go to infinity, and the first inequality forces the partial sums of the series to go to infinity as well, so the series is divergent. If the integral converges, then the second inequality puts an upper bound on the partial sums of the series, and since they are a nondecreasing sequence, they must converge to a finite sum for the series. (See the explanation preceding Exercises 61 and 62 in Section 10.4.)

53.



Comparing areas in the figures, we have for all $n \ge N$, $\int_N^{n+1} f(x)\,dx < a_N + \cdots + a_n$

$$< a_N + \int_N^n f(x)\,dx.$$

If the integral diverges, it must go to infinity, and the first inequality forces the partial sums of the series to go to infinity as well, so the series is divergent. If the integral converges, then the second inequality puts an upper bound on the partial sums of the series, and since they are a nondecreasing sequence, they must converge to a finite sum for the series. (See the explanation preceding Exercises 61 and 62 in Section 10.4.)

54. **(a)** Diverges by the Limit Comparison Test.

Let $a_k = \frac{1}{\sqrt{2k+7}}$ and $b_k = \frac{1}{k^{1/2}}$ Then $a_k > 0$ and $b_k > 0$ for $k \ge 1$ and

$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{k^{1/2}}{\sqrt{2k+7}} = \frac{1}{\sqrt{2}}$. Since $\sum_{k=1}^{\infty} b_k$ diverges by the p-Test with $p=\frac{1}{2}$, $\sum_{k=1}^{\infty} a_k$ also diverges.

(b) Diverges by the nth-Term Test, since

$$\lim_{k\to\infty} \left(1+\frac{1}{k}\right)^k = e \ne 0.$$

(c) Converges absolutely by the Direct Comparison Test, since $\left|\dfrac{\cos k}{k^2+\sqrt{k}}\right| < \dfrac{1}{k^2}$ for $k \ge 1$ and $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges by the p-Test with $p = 2$.

(d) Diverges by the Integral Test, since $\displaystyle\int_3^{\infty} \frac{18}{x \ln x}\,dx = \lim_{b\to\infty} \Big[18 \ln|\ln x|\Big]_3^b = \infty.$

55. $\displaystyle\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n^n |x+2|^n}{3^n n!}$, use the Ratio Test.

$$\lim_{n\to\infty} \frac{(n+1)^{n+1}|x+2|^{n+1}}{3^{n+1}(n+1)!} \cdot \frac{3^n n!}{n^n |x+2|^n}$$
$$= \frac{|x+2|}{3} \lim_{n\to\infty} \frac{(n+1)^n}{n^n}$$
$$= \frac{|x+2|}{3} \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$$
$$= \frac{|x+2|}{3} \cdot e$$
$$= |x+2| \cdot \frac{e}{3}$$

The series converges absolutely for $|x+2| < \dfrac{3}{e}$ and diverges for $|x+2| > \dfrac{3}{e}$. Therefore, the radius of convergence is $\dfrac{3}{e}$.

56. $\displaystyle\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n!|x|^n}{n^n 5^n}$, use the Ratio Test.

$$\lim_{n\to\infty} \frac{(n+1)!|x|^{n+1}}{(n+1)^{n+1} 5^{n+1}} \cdot \frac{n^n 5^n}{n!|x|^n}$$
$$= \frac{|x|}{5} \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$$
$$= \frac{|x|}{5} \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n}$$
$$= \frac{|x|}{5e}$$

The series converges absolutely for $|x| < 5e$, diverges for $|x| > 5e$. Therefore, the radius of convergence is $5e$.

57. One possible answer: $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \ln n}$

This series diverges by the integral test, since $\displaystyle\int_3^{\infty} \frac{1}{x \ln x}\,dx = \lim_{b\to\infty} \Big[\ln|\ln x|\Big]_3^b = \infty.$ Its partial sums are roughly $\ln(\ln n)$, so they are much smaller than the partial sums for the harmonic series, which are about $\ln n$.

58. (a)
$$a_k = (-1)^{k+1} \int_0^{1/k} 6(kx)^2\,dx$$
$$= (-1)^{k+1}\Big[2k^2x^3\Big]_0^{1/k}$$
$$= (-1)^{k+1}\left(\frac{2}{k}\right)$$

(b) The series converges by the Alternating Series Test.

(c) The first few partial sums are:

$S_1 = 2,\ S_2 = 1,\ S_3 = \dfrac{5}{3},\ S_4 = \dfrac{7}{6},$

$S_5 = \dfrac{47}{30},\ S_6 = \dfrac{37}{30},\ S_7 = \dfrac{319}{210},$

$S_8 = \dfrac{533}{420},\ S_9 = \dfrac{1879}{1260}.$ For an alternating series, the sum is between any two adjacent partial sums, so

$1 < S_8 \le \text{sum} \le S_9 < \dfrac{3}{2}.$

59. (a) Diverges by the Limit Comparison Test.

Let $a_n = \dfrac{n}{3n^2+1}$ and $b_n = \dfrac{1}{n}$. Then $a_n > 0$ and $b_n > 0$ for $n \ge 1$, and $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{3n^2+1} = \frac{1}{3}.$ Since $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges, $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

(b) $\displaystyle S = \sum_{n=1}^{\infty} \frac{n}{3n^2+1} \cdot \frac{3}{n} = \sum_{n=1}^{\infty} \frac{3}{3n^2+1}.$

This series converges by the Direct Comparison Test, since $\dfrac{3}{3n^2+1} < \dfrac{1}{n^2}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-Test with $p = 2$.

60. **(a)** From the list of Maclaurin series in Section 10.2,

$\ln(1+x)$
$= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots.$

(b) $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x|^{n+1}}{n+1}\cdot\frac{n}{|x|^n} = |x|\lim_{n\to\infty}\frac{n}{n+1} = |x|$$

The series converges for $|x| < 1$ and diverges for $|x| > 1$.

Check $x = 1$: $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ converges by the Alternating Series Test.

Check $x = -1$: $-\sum_{n=1}^{\infty}\frac{1}{n}$ diverges by the p-Test. The series converges for $-1 < x \le 1$.

(c) To estimate $\ln\frac{3}{2}$, we would let $x = \frac{1}{2}$. The truncation error is less than the magnitude of the sixth nonzero term, or

$$\left|-\frac{x^6}{6}\right| = \frac{1}{2^6\cdot 6} = \frac{1}{384} < 0.002605.$$

Thus, a bound for the (absolute) truncation error is 0.002605.

(d) $$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{2n}}{2n} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x^2)^n}{n} = \frac{1}{2}\ln(1+x^2)$$

61. $$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{2^{k+1}|x|^{k+1}}{\ln(k+3)}\cdot\frac{\ln(k+2)}{2^k|x|^k} = 2|x|$$

The series converges absolutely for $|x| < \frac{1}{2}$, or $-\frac{1}{2} < x < \frac{1}{2}$.

Check $x = -\frac{1}{2}$: $\sum_{k=0}^{\infty}\frac{(-1)^k}{\ln(k+2)}$ converges by the Alternating Series Test.

Check $x = \frac{1}{2}$: $\sum_{k=0}^{\infty}\frac{1}{\ln(k+2)}$ diverges by the Direct Comparison Test, since $\frac{1}{\ln(k+2)} > \frac{1}{k}$ for all $k \ge 2$, and $\sum_{k=1}^{\infty}\frac{1}{k}$ diverges (harmonic).

The original series converges for $-\frac{1}{2} \le x < \frac{1}{2}$.

62. **(a)** The series converges by the Direct Comparison Test, since $\frac{1}{n^p\ln n} < \frac{1}{n^p}$ for $n \ge 3$, and $\sum_{n=3}^{\infty}\frac{1}{n^p}$ converges as a p-series when $p > 1$.

(b) For $p = 1$, the series is $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$, which diverges by the Integral Test, since

$$\int_2^{\infty}\frac{1}{x\ln x}\,dx = \lim_{b\to\infty}[\ln(\ln x)]_2^b = \infty.$$

(c) For $0 \le p < 1$, we have $\frac{1}{n^p\ln n} > \frac{1}{n\ln n}$, so $\sum_{n=2}^{\infty}\frac{1}{n^p\ln n}$ diverges by the Direct Comparison Test with $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ from part (b).

63. $\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$, so at $x = 1$ the series is $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$. This series converges by the Alternating Series Test.

64. $\arctan x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$

At $x = -1$, the series is

$\sum_{n=0}^{\infty}\frac{(-1)^n(-1)^{2n+1}}{2n+1} = -\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$, which converges by the Alternating Series Test. At $x = 1$, the series is $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$, which converges by the Alternating Series Test.

65. **(a)** It fails to satisfy $u_n \ge u_{n+1}$ for all $n \ge N$.

(b) The sum is

$$\left(\sum_{n=1}^{\infty}\frac{1}{3^n}\right)-\left(\sum_{n=1}^{\infty}\frac{1}{2^n}\right)=\frac{\frac{1}{3}}{1-\frac{1}{3}}-\frac{\frac{1}{2}}{1-\frac{1}{2}}=\frac{1}{2}-1=-\frac{1}{2}.$$

66. True, $\sum_{n=1}^{\infty}|a_n|=\sum_{n=1}^{\infty}\frac{|x|^{2n}}{2n}$

Use the Ratio Test to find the endpoints.

$$\lim_{n\to\infty}\frac{|x|^{2(n+1)}}{2(n+1)}\cdot\frac{2n}{|x|^{2n}}=|x|^2\lim_{n\to\infty}\frac{n}{n+1}=|x|^2$$

The series converges absolutely for $|x|^2<1,$ or $|x|<1.$

The endpoints are $x=\pm1$. At both endpoints the series equals: $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n},$ which converges by the Alternating Series Test.

67. True; the next term is $a_{101}=\frac{(-1)^{101}}{101^2},$ which is negative, so s_{100} must be greater than the sum of the series.

68. B; $\sum_{n=1}^{\infty}|a_n|=\sum_{n=1}^{\infty}\frac{n|2x-5|^n}{n+2},$ use the Ratio Test.

$$\lim_{n\to\infty}\frac{(n+1)|2x-5|^{n+1}}{n+3}\cdot\frac{n+2}{n|2x-5|^n}$$
$$=|2x-5|\lim_{n\to\infty}\frac{n^2+3n+2}{n^2+3n}$$
$$=|2x-5|$$

The series converges absolutely when $|2x-5|<1,$ or $\left|x-\frac{5}{2}\right|<\frac{1}{2}.$

69. A; the series converges absolutely for $\left|x-\frac{5}{2}\right|<\frac{1}{2};$ the series diverges at the endpoints by the nth-Term Test. The interval of convergence is: $2<x<3$.

70. E

I. $4\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges by the p-Test.

II. $\sum_{n=1}^{\infty}\left(\frac{1}{\ln 4}\right)^n$ converges (geometric series with $r\approx 0.7$).

III. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$ converges absolutely by the p-Test (or use the Alternating Series Test).

71. B; the truncation error is

$$\left|\sum_{n=101}^{\infty}\frac{(-1)^n}{2^n}\right|=\left|\sum_{n=101}^{\infty}-\left(\frac{1}{2}\right)^n\right|.$$

Notice that $\sum_{n=101}^{\infty}\left(-\frac{1}{2}\right)^n$ is a geometric series with first term $\left(-\frac{1}{2}\right)^{101}$ and constant ratio $r=-\frac{1}{2}$. It converges to

$$\frac{\left(-\frac{1}{2}\right)^{101}}{1-\left(-\frac{1}{2}\right)}=-\frac{\left(\frac{1}{2}\right)^{101}}{\frac{3}{2}}=-\frac{1}{3\cdot 2^{100}}.$$

Thus $\frac{1}{3\cdot 2^{100}}$ is the truncation error.

72. Answers will vary.

73. **(a)** $\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\sqrt[n]{\frac{n^2}{2^n}}=\lim_{n\to\infty}\frac{\left(\sqrt[n]{n}\right)^2}{2}=\frac{1}{2}$

The series converges.

(b)
$$\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\sqrt[n]{\left(\frac{n}{2n-1}\right)^n}=\lim_{n\to\infty}\frac{n}{2n-1}=\frac{1}{2}$$

The series converges.

(c) $\lim\limits_{n\to\infty,\, n \text{ odd}} \sqrt[n]{a_n} = \lim\limits_{n\to\infty,\, n \text{ odd}} \sqrt[n]{\dfrac{n}{2^n}}$

$= \lim\limits_{n\to\infty,\, n \text{ odd}} \dfrac{\sqrt[n]{n}}{2}$

$= \dfrac{1}{2}$

$\lim\limits_{n\to\infty,\, n \text{ even}} \sqrt[n]{a_n} = \lim\limits_{n\to\infty,\, n \text{ even}} \sqrt[n]{\dfrac{1}{2^n}} = \dfrac{1}{2}$

Thus, $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \dfrac{1}{2}$, so the series converges.

74. **(a)** $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} \sqrt[n]{\dfrac{|x-1|^n}{4n}} = \dfrac{|x-1|}{4}$.

The series converges absolutely if $\dfrac{|x-1|}{4} < 1$, or $-3 < x < 5$.

Check $x = -3$: $\sum\limits_{n=0}^{\infty} (-1)^n$ diverges.

Check $x = 5$: $\sum\limits_{n=0}^{\infty} 1^n$ diverges.

The interval of convergence is $(-3, 5)$.

(b) $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} \sqrt[n]{\dfrac{|x-2|^n}{n \cdot 3^n}}$

$= \lim\limits_{n\to\infty} \dfrac{|x-2|}{\sqrt[n]{n} \cdot 3}$

$= \dfrac{|x-2|}{3}$.

The series converges absolutely if $\dfrac{|x-2|}{3} < 1$, or $-1 < x < 5$.

Check $x = -1$: $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{n}$ converges.

Check $x = 5$: $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ diverges.

The interval of convergence is $[-1, 5)$.

(c) $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} \sqrt[n]{2^n |x|^n} = 2|x|$. The series converges absolutely if $2|x| < 1$, or $-\dfrac{1}{2} < x < \dfrac{1}{2}$.

Check $x = -\dfrac{1}{2}$: $\sum\limits_{n=1}^{\infty} (-1)^n$ diverges.

Check $x = \dfrac{1}{2}$: $\sum\limits_{n=1}^{\infty} 1$ diverges.

The interval of convergence is $\left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$.

(d) $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} \sqrt[n]{|\ln x|^n}$

$= |\ln x|$.

The series converges absolutely if $|\ln x| < 1$, or $\dfrac{1}{e} < x < e$.

Check: $x = \dfrac{1}{e}$: $\sum\limits_{n=0}^{\infty} \left(\ln \dfrac{1}{e}\right)^n = \sum\limits_{n=0}^{\infty} (-1)^n$ diverges.

Check $x = e$: $\sum\limits_{n=0}^{\infty} (\ln e)^n = \sum\limits_{n=0}^{\infty} 1^n$ diverges.

The interval of convergence is $\left(\dfrac{1}{e}, e\right)$.

Quick Quiz Sections 10.4 and 10.5

1. E

I. $\sum\limits_{n=1}^{\infty} \dfrac{2}{n^2+1}$ converges by the Limit Comparison Test $\left(\text{Use } \sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}, \text{ which converges by the } p\text{-Test.}\right)$

II. $\sum\limits_{n=1}^{\infty} \dfrac{2^n - 1}{3^n + 1}$ converges by the Limit Comparison Test $\left(\text{Use } \sum\limits_{n=1}^{\infty} \left(\dfrac{2}{3}\right)^n, \text{ geometric series with } r = \dfrac{2}{3}.\right)$

III. $\sum\limits_{n=1}^{\infty} \dfrac{\sqrt[4]{n}}{n} = \sum\limits_{n=1}^{\infty} \dfrac{1}{n^{3/4}}$ diverges by the p-Test.